GROUPS OF HARMONIC TRANSFORMATIONS

By Edward Kasner and John DeCicco

1. **The infinite set (H) of harmonic transformations.** Consider a transformation T of the real or complex cartesian plane,

$$X = \phi(x, y), \qquad Y = \psi(x, y), \tag{1}$$

with jacobian,

$$J = \phi_x\psi_y - \phi_y\psi_x \neq 0, \tag{2}$$

such that the components ϕ and ψ satisfy the Laplace equation,

$$\phi_{xx} + \phi_{yy} = 0, \qquad \psi_{xx} + \psi_{yy} = 0, \tag{3}$$

in a certain region of the (x, y)-plane. We shall term any such transformation T a *harmonic correspondence.*

Harmonic transformations should not be confused with conformal maps. In general, the components of a harmonic transformation are not interrelated in any way whatsoever. The components ϕ and ψ of a conformal correspondence are of course conjugate-harmonic, that is, they satisfy the direct or reverse Cauchy-Riemann equations,

$$\phi_x = \pm\psi_y\,, \qquad \phi_y = \mp\psi_x\,. \tag{4}$$

Thus a harmonic transformation is conformal if and only if its components are conjugate-harmonic.

General harmonic transformations are of interest in connection with the theory of minimal surfaces. Harmonic functions and, incidentally, harmonic transformations appear in the theory of minimal surfaces and the Plateau problem. In particular, see the fundamental papers of Schwarz and Douglas. Schwarz discusses the case where the jacobian of a harmonic transformation vanishes, and studies possible singularities. Douglas studies the inverses of harmonic transformations. (See [2], [3] and [13].)

All harmonic correspondences form an infinite set (H) of $\infty^{4f(1)}$ transformations since they are defined by essentially four independent functions of a single variable. The totality of harmonic transformations of course do not constitute a group. The conformal group consists of $\infty^{2f(1)}$ transformations and is a subset of our set (H) of harmonic correspondences.

In the present paper, we shall discuss the groups contained in our infinite set (H) of harmonic transformations. Our procedure is to obtain in the real and imaginary domains all the harmonic transformations whose inverses are harmonic. From this result, we shall deduce after a long discussion the groups of harmonic transformations.

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