THE L^2-SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

By Philip Hartman

1. Let $f(t)$ and $g(t)$ be real-valued, continuous functions on the half-axis $0 \leq t < \infty$, satisfying the respective conditions

(i) $f(t)$ is bounded from above, and
(ii) $g(t)$ is of class L^2, that is, $\int^\infty g^2(t)\, dt < \infty$.

Wintner [1] has shown that if $x = x(t)$, $0 \leq t < \infty$, is a solution of the non-homogeneous linear differential equation

$$x'' + f(t)x = g(t) \tag{1}$$

and is of class L^2,

$$\int^\infty x^2(t)\, dt < \infty, \tag{2}$$

then $x'(t)$ is of class L^2,

$$\int^\infty x'^2(t)\, dt < \infty \tag{3}$$

and

$$x(t) \to 0 \qquad (t \to \infty). \tag{4}$$

In addition, he has shown that if the condition (i) is changed to

(i bis) $f(t)$ is bounded,

then, also,

$$x'(t) \to 0 \qquad (t \to \infty). \tag{5}$$

(Actually, this last statement was formulated only for the homogeneous case, $g(t) \equiv 0$, of (2),

$$x'' + f(t)x = 0, \tag{6}$$

but the given proof is valid whenever (ii) holds.) Wintner [1] raises the question as to whether or not (5) can be asserted without changing (i) to (i bis). The object of this note is to answer the question in the affirmative.

(I) *Let* $f(t)$ *and* $g(t)$, $0 \leq t < \infty$, *be real-valued, continuous functions satisfying* (i) *and* (ii), *respectively. Let* $x = x(t)$, $0 \leq t < \infty$, *be a solution of* (1) *satisfying* (2). *Then* (5) *holds* (*as do* (3) *and* (4)).

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