A PROBLEM OF ARRANGEMENTS

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1. Introduction.

1.1. J. Bertrand [6] announced in 1887 the following result:

If, in a ballot, candidate P scores p votes and candidate Q scores $q < p$ *votes, then the probability that throughout the counting there are always more votes for P than for Q equals* $(p - q)/(p + q)$.

Bertrand merely sketched a method of proof, which proceeded by induction, and expressed the opinion that a straightforward proof might be devised.

This was supplied, in the same year, by D. André [4]. Calling *unfavorable* an arrangement of votes for which P does not lead throughout, André established a one-to-one correspondence between the unfavorable arrangements beginning with a vote for P and all the arrangements beginning with a vote for Q. Now all the latter are unfavorable, and their probability is $q/(p + q)$; hence the probability sought by Bertrand is $1 - 2q/(p + q) = (p - q)/(p + q)$.

André's proof or variations of it may be found in most of the classical treatises on the theory of probability (*e.g.* those of Bertrand, Poincaré, Borel and Czuber. The proofs of Lucas [11] and J. Aebli [1] are mere variations. See also D. Mirimanoff [12]).

1.2. Bertrand's result was generalized by É. Barbier [5]. Barbier's statement is rather vague, but he is credited by Lazarus [10; 201] with the following:

If $\alpha \geq 0$ *is an integer and* $p > \alpha q$, *then the probability that, throughout the counting, the number of votes registered for P is always greater than* α *times the number registered for Q equals* $(p - \alpha q)/(p + q)$.

Barbier does not even hint at his method of proof. Presumably, since his note preceded André's, he used Bertrand's method. Later on, in 1923, Barbier's result was rediscovered by A. Aeppli [2]. (This dissertation was not available to us and we derive our knowledge from a comment of A. Aeppli [3] on the paper of Aebli [1].)

1.3. In this paper we start by giving a new direct proof of Bertrand's and Barbier's results. We actually prove a stronger theorem (Theorem 1) since, breaking up the $(p + q)!$ sequences in which the $p + q$ individual votes can be arranged into $(p + q - 1)!$ groups of $p + q$ sequences each, we show that within every such group there are $p - \alpha q$ "favorable" and $(\alpha + 1)q$ "unfavorable" sequences. Every group consists of the $p + q$ sequences obtained from a given sequence by cyclic permutations.

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