THE RIEMANN FUNCTION FOR $\partial^2 u/\partial x \partial y + H(x + y)u = 0$

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1. Introduction. Some interesting hyperbolic differential equations can be represented in the form

$$\partial^2 u/\partial x\, \partial y + H(x + y)u = 0. \tag{1}$$

An instance of (1) as the wave equation in space was first considered by Euler [5; 480–81]. Instances of this equation in the field of differential geometry (Laplace) are given in Darboux [3; 52], and instances in the field of compressible hydrodynamics were considered by Riemann [6; 173] for linear flows and by A. Steichen (see Ackaret [1; 317]) for plane flows.

We shall concern ourselves entirely with the evaluation of the *Riemann function* of (1), by means of which Riemann showed how to solve (1) for given boundary conditions. The Riemann function, despite its crucial importance, seems to have been found in *closed form* only in the special cases $H(r) = -\lambda$ and $H(r) = -\lambda(\lambda + 1)/r^2$, λ constant. Moreover these cases are just those solved by Riemann, who, in turn, has even been anticipated to some extent by Euler [4]! Undoubtedly, the *Picard iteration process* and its existence proof for the Riemann function in general, drew attention away from the evaluation of this function. Nevertheless, the fact that the Riemann function has been evaluated only for these two simple cases restricts the number of physical situations to which equation (1) can correspond and still remain simple enough to permit solution in closed form by use of a Riemann function.

It will therefore be our purpose to consider the possibility of finding the Riemann function for a greater variety of functions $H(r)$. The methods will be twofold. First of all we shall use a specialized artifice to obtain the Riemann function in closed form for a few types of $H(r)$, namely $H(r) = -\lambda(\lambda + 1)\mu^2/\sinh^2 \mu(r + \nu)$ (λ, μ, ν constant), and some of its limiting cases, $H(r) = -\lambda \exp \mu r$, $-\lambda(\lambda + 1)/r^2$, and $-\lambda$. Then we shall consider a certain *Fourier integral* expression for the Riemann function of (1) with arbitrary $H(r)$. This latter expression is due to Riemann, who derived it in a *heuristic* manner. We shall also indicate a rigorous proof of Riemann's expression, since this long-neglected expression is of potential importance with the further development of analysis.

2. Some properties of the Riemann function. For later reference we shall summarize some existence and uniqueness theorems as obtained through the Picard iteration process (see Courant-Hilbert [2; 317–322]). For equation (1),

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