A PROBLEM IN CONNECTED FINITE CLOSURE ALGEBRAS

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1. **Introduction.** In answer to a question left open by J. C. C. McKinsey and Alfred Tarski [2; 160], it is shown in this paper that the closure algebra over a Euclidean space is a universal algebra for the class of all connected finite closure algebras. The solution of the problem is made to depend on the existence of certain interior transformations, and the greater part of the paper is devoted to the construction of these transformations.

Capital Roman letters will denote either elements of an algebra or sets of points. Lower-case Roman letters will denote integers when used as subscripts; otherwise they denote either points of a space or sets consisting of single points. Whether X and Y are elements of an algebra or set, their union and intersection will be denoted by $X \cup Y$ and $X \cap Y$, respectively. In either case X contained in Y will be written $X \subset Y$, and X' will designate the complement of X. Finally, $\overline{X}$ will denote the closure of X in either the algebraic or topological sense. It will be clear from the context whether the symbols have a set-theoretic or algebraic meaning.

Topological space is used in the sense of Alexandroff-Hopf [1; 37] and is what McKinsey and Tarski call a topological space in the wider sense.

2. **Closure algebras and interior transformations.** Following McKinsey and Tarski, a *closure algebra* is defined as follows:

Definition 2.1. A set K is a *closure algebra* with respect to the operations $\cup$, $\cap$, $'$, and $^-$, when

(a) K is a Boolean algebra with respect to $\cup$, $\cap$, and $'$.
(b) If X is in K, then $X \subset \overline{X} \subset K$.
(c) If X is in K, then $\overline{\overline{X}} = \overline{X}$.
(d) If X and Y are in K, then $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$.
(e) The closure of the zero element is the zero element; i.e., $\overline{0} = 0$.

The *closure algebra over a topological space* S has for its elements the subsets of S, which become a closure algebra under the operations of set-theoretic union, intersection and complementation, and topological closure.

A closure algebra is *connected* if $\overline{X} \cap \overline{X'} = 0$ implies either $X = 0$ or $X = I$, where I is the unit element of the Boolean algebra.

The relation between abstract finite closure algebras and closure algebras over finite topological spaces is contained in the following

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