REPRESENTATION OF *-ALGEBRAS

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1. **Introduction.** A *Banach algebra* A is a Banach space in which, besides the scalar multiplication and addition, there is defined a continuous multiplication, so that the elements form a ring, or rather a linear algebra.

A *Banach *-algebra* A is a Banach algebra with complex scalars $\{\lambda, \mu, \cdots\}$ which has defined in it a semi-linear operator (*), satisfying

$$(\lambda f + \mu g)^* = \bar{\lambda} f^* + \bar{\mu} g^* \qquad (f, g \in A)$$

$$(fg)^* = g^* f^*$$

$$f^{**} = f.$$

We shall represent those commutative Banach *-algebras which satisfy the condition

$$\text{C}^*: \qquad k \, \| f \| \, \| f^* \| \leq \| f f^* \|$$

for every $f \in A$, k being a positive real number independent of f. We state the result for the case that A has a unit: A can be represented as the class of all continuous complex-valued functions on a suitable compact Hausdorff space X such that $\| \quad \|'$ defined by

$$\| f \|' = \sup_{x \in X} | f(x) |$$

is a norm equivalent to the original norm of A; and the *-operation is represented by

$$(1) \qquad f^*(x) = \overline{f(x)} \qquad (f \in A, x \in X).$$

When A has no unit, X is locally compact, and the functions f all "vanish at infinity". This case is reduced to the previous one in Lemma 4 by introducing a unit but retaining C*.

Our treatment begins by establishing (1) by means of the following (Lemma 3):

*Let A be a Banach *-algebra satisfying* C* *and also*

$$\text{M}: \qquad \| fg \| \leq \| f \| \, \| g \| \qquad (f, g \in A).$$

Let f be such that $ff^* = f^*f$, and write $f = u + iv$, $u = u^*$, $v = v^*$. Then any complex number $x + iy$ in the spectrum of f satisfies

$$| x \cos \theta + y \sin \theta | \leq \| u \cos \theta + v \sin \theta \|$$

for any value of the angle θ.

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