A THEOREM OF LICHTENSTEIN

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In the present note the following theorem of Lichtenstein [8; 34–42] is generalized: Let the function $\rho(x, y)$ be quadratically integrable in a rectangle $\mathfrak{R}$ in the (x, y)-plane, then the function

$$(1)_2 \qquad \phi(x, y) = \frac{1}{2\pi} \int\int_{\mathfrak{R}} (\log r)\rho(x', y')\, dx'\, dy',$$

or using an operator symbol P

$$\phi(x, y) = P\rho(x, y),$$

possesses almost everywhere second derivatives ϕ_{xx}, ϕ_{xy}, ϕ_{yy}, which are quadratically integrable over any bounded domain. Here $r^2 = (x - x')^2 + (y - y')^2$. It is in particular true that ϕ admits almost everywhere the Laplacian operator Δ and that almost everywhere $\Delta\phi = \phi_{xx} + \phi_{yy} = \rho$.

Precisely, Lichtenstein's statement is this: For almost all y, ϕ possesses a derivative ϕ_x which in its turn is absolutely continuous in x and possesses for almost all x a derivative ϕ_{xx}. Further, for an appropriate set $\mathfrak{O}$ of measure zero, the derivative ϕ_x is defined in $\mathfrak{R} - \mathfrak{O}$ and possesses there a derivative ϕ_{xy} with respect to y provided that in forming this derivative only points x, y of the set $\mathfrak{R} - \mathfrak{O}$ are employed. The same is true for y instead of x. Almost everywhere $\phi_{xx} + \phi_{yy} = \rho$ and $\phi_{xy} = \phi_{yx}$. All derivatives ϕ_x, ϕ_y, ϕ_{xx}, ϕ_{xy}, ϕ_{yx}, ϕ_{yy} are quadratically integrable with respect to x and y.

For our generalization of this theorem to the case of $N \geq 2$ independent variables $x = \{x_1, \cdots, x_N\}$, we must define the operation

$$(1)_N \qquad \phi(x) = P\rho(x) = -\Omega_N^{-1} \int_{\mathfrak{R}} r^{-N+2}\rho(\bar{x})\, d\bar{x} \qquad (N > 2),$$

for quadratically integrable ρ, where $\mathfrak{R}$ is an open bounded region, $r^2 = (x_1 - \bar{x}_1)^2 + \cdots + (x_N - \bar{x}_N)^2$, and Ω_N is the content of the N-dimensional unit-sphere. It is known that $\phi(x)$ is continuous if ρ is continuously differentiable. If a quadratically integrable function ρ is approximated in the mean by continuously differentiable functions $\rho^\star$ then, as will be shown, the functions $\phi^\star = P\rho^\star$ approach in the mean a quadratically integrable limit function ϕ. The operation P is then defined through $P\rho = \phi$.

For the function ϕ defined by $(1)_N$, $(N \geq 2)$, we state

Theorem 1: *The function $\phi = P\rho$ for quadratically integrable ρ agrees almost everywhere with a function ϕ^* which for almost all $\{x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_N\}$*

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