THE UNIFORM APPROXIMATION TO CONTINUOUS FUNCTIONS BY LINEAR AGGREGATES OF FUNCTIONS OF A GIVEN SET

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Introduction. The following paragraphs are but slightly connected, and in fact form a compilation of notes rather than one single paper. A certain degree of unity arises, however, from the repeated use of the following fundamental theorem of F. Riesz [12].

Theorem A. *It will be possible to approximate uniformly over* $[a, b]$ *to a function* $h(x) \in C$, *the space of functions continuous on the closed interval* $[a, b]$, *by linear aggregates of functions of a set* $\{\phi(x)\} \in C$ *when, and only when, the equations*

$$\int_a^b \psi(x)\, dg(x) = 0 \tag{1}$$

(for all $\psi(x) \in \{\phi(x)\}$*) imply*

$$\int_a^b h(x)\, dg(x) = 0,$$

whenever $g(x)$ *is a function of bounded variation over* $[a, b]$.

The set $\{\phi(x)\}$ of functions not necessarily belonging to C is said to be *closed* in C if it is possible to approximate uniformly over $[a, b]$ to *all* functions $h(x) \in C$ by linear aggregates of functions $\in \{\phi(x)\}$. More generally, a set of elements $\{\phi\}$ (not necessarily) belonging to a metric space R is said to be closed in R if the common part of R and the set of all linear aggregates of elements of $\{\phi\}$ is dense everywhere in R. (See [6].) The distance of two functions $f_1(x), f_2(x) \in C$ is given by

$$\max_{a \le x \le b} | f_1(x) - f_2(x) | ;$$

in $L_p(a, b)$, the space of all measurable functions $f(x)$ for which

$$\int_a^b | f(x)|^p \, dx$$

is finite, the distance of $f_1(x)$ and $f_2(x)$ is usually defined by

$$\left\{\int_a^b | f_1(x) - f_2(x) |^p \, dx\right\}^{1/p}.$$

By Theorem A *a set* $\{\phi(x)\} \in C$ *is closed when, and only when, the equations* (1) *imply* $g(a) = g(x) = g(b)$ *at all points of continuity of* $g(x)$.

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