LIMITS FOR THE CHARACTERISTIC ROOTS OF A MATRIX. II.

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This paper is a continuation of my paper "Limits for the characteristic roots of a matrix", this Journal, vol. 13 (1946), pp. 387–395. I am able now to improve some of the results obtained in this paper.

We continue the numeration of the theorems and equations. Again, let $A = (a_{\kappa\lambda})$ be a square matrix of order n, and ω an arbitrary characteristic root of A. We set

$$\sum_{\substack{\nu=1 \\ \nu\neq\kappa}}^{n} | a_{\kappa\nu} | = P_\kappa \qquad \text{and} \qquad \sum_{\substack{\nu=1 \\ \nu\neq\kappa}}^{n} | a_{\nu\kappa} | = Q_\kappa .$$

Theorem 1 will be improved as follows.

Each characteristic root ω of A lies in the interior or on the boundary of at least one of the $n(n-1)/2$ ovals of Cassini

$$| z - a_{\kappa\kappa} || z - a_{\lambda\lambda} | \leq P_\kappa P_\lambda \qquad (\kappa \neq \lambda), \tag{21}$$

and in at least one of the $n(n-1)/2$ ovals

$$| z - a_{\kappa\kappa} || z - a_{\lambda\lambda} | \leq Q_\kappa Q_\lambda \qquad (\kappa \neq \lambda).$$

This is sharper than Theorem 1 since every point of the oval (21) lies in the interior or on the boundary of at least one of the circles $| z - a_{\kappa\kappa} | = P_\kappa$ and $| z - a_{\lambda\lambda} | = P_\lambda$.

We set

$$(22)\qquad \begin{aligned} M &= \tfrac{1}{2} \max_{\substack{\kappa,\lambda=1,2,\cdots,n \\ \kappa\neq\lambda}} \{ | a_{\kappa\kappa} | + | a_{\lambda\lambda} | + ((| a_{\kappa\kappa} | - | a_{\lambda\lambda} |)^2 + 4P_\kappa P_\lambda)^{\frac{1}{2}} \}, \\ m &= \tfrac{1}{2} \min_{\substack{\kappa,\lambda=1,2,\cdots,n \\ \kappa\neq\lambda}} \{ | a_{\kappa\kappa} | + | a_{\lambda\lambda} | - ((| a_{\kappa\kappa} | - | a_{\lambda\lambda} |)^2 + 4P_\kappa P_\lambda)^{\frac{1}{2}} \}. \end{aligned}$$

It follows from (21) that

$$| \omega | \leq M. \tag{23}$$

This is sharper than Theorem 2 if

$$\max_{\nu=1,2,\cdots,n} \{ | a_{\nu\nu} | + P_\nu \} = | a_{kk} | + P_k > \max_{\substack{\nu=1,2,\cdots,n \\ \nu\neq k}} \{ | a_{\nu\nu} | + P_\nu \}.$$

Denote the determinant of A by D. It follows from (23) that $| D | \leq M^n$.

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