THE THEOREM OF MINKOWSKI-HLAWKA

By Kurt Mahler

Let R_n, where $n \geq 2$, be the n-dimensional Euclidean space of all points $X = (x_1, \cdots, x_n)$ with real coordinates. A symmetrical bounded star body K in R_n is defined as a closed bounded point set containing the origin $O = (0, \cdots, 0)$ as an inner point and bounded by a continuous surface C symmetrical in O which meets every radius vector from O in just one point. A lattice

$$\Lambda: x_h = \sum_{k=1}^{n} a_{hk}u_k \qquad (h = 1, 2, \cdots, n; u_1, \cdots, u_n = 0, \mp 1, \mp 2, \cdots)$$

of determinant

$$d(\Lambda) = \left| \, | a_{hk} |_{h,k=1,2,\ldots,n} \right|$$

is called K-admissible if no point of Λ except O is an inner point of K. Denote by

$$V(K) = \int_K \cdots \int dx_1 \cdots dx_n$$

the volume of K, by $\Delta(K)$ the lower bound of $d(\Lambda)$ extended over all K-admissible lattices, and put

$$Q(K) = \frac{V(K)}{\Delta(K)}.$$

A critical lattice of K is defined as a K-admissible lattice Λ such that $d(\Lambda) = \Delta(K)$.

A theorem due to Minkowski [4; 265, 270, 277], but first proved by E. Hlawka [2; 288–298] and C. L. Siegel [6], states that

$$\text{(a)} \qquad Q(K) \geq 2\zeta(n) \qquad \left(\zeta(n) = \sum_{v=1}^{\infty} v^{-n}\right)$$

for all symmetrical bounded star bodies. It is a difficult problem to decide whether the constant on the right-hand side is the best possible one. In the present note, K is assumed to be a *symmetrical convex body*; under this restriction, the constant $2\zeta(n)$ in (a) will be shown to be replaceable by a larger number.

The method used is quite different from that of Hlawka and Siegel, and depends essentially on the theorem of Brunn and Minkowski on the sections of a convex body. (See [1; §48].)

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