SOME THEOREMS ON EINSTEIN 4-SPACE

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1. An *Einstein space* is a Riemannian space with a metric $d^*s^2 = {}^*g_{\kappa\lambda}\, dx^\kappa\, dx^\lambda$ ($\kappa, \lambda = 1, 2, \cdots, n$) whose Ricci tensor ${}^*R_{\kappa\lambda}$ satisfies the equation

$$^*R_{\kappa\lambda} = c\, {}^*g_{\kappa\lambda}\,,$$

where c is a scalar. (In general we follow Eisenhart's notations and conventions [2].) The main result of this paper is the following generalization of a theorem due to Slebodzinski [5; Chap. 1]:

Theorem 1.1. *Let* $g_{ij}(x^k)$ $(h, i, j, k, l = 1, 2, 3)$ *be a positive definite tensor with two and only two equal Ricci invariants. Then in order that there exist Einstein 4-spaces with a metric of the form*

$$d^*s^2 = [\rho(x^k, t)]^2\, dt^2 - g_{ij}(x^k)\, dx^i\, dx^j, \tag{1.1}$$

or

$$d^*s^2 = [\sigma(x^k, t)]^{-2}\, dt^2 - [\rho(x^k, t)]^{-2}\, g_{ij}(x^k)\, dx^i\, dx^j \qquad (\partial_t \rho \neq 0), \tag{1.2}$$

it is necessary and sufficient that the scalar curvature of g_{ij} *is constant*:

$$a = -R/6 = \text{const.}, \tag{1.3}$$

and that there exists an orthogonal ennuple $v^k_{(\alpha)}$ $(\alpha, \beta, \gamma, \delta, \epsilon = 1, 2, 3)$ *of Ricci congruences whose coefficients of rotation satisfy the following conditions*:

$$\begin{gathered} \gamma_{312} = \gamma_{321} = \gamma_{123} = 0, \\ \gamma_{311} = \gamma_{322}\,, \\ \gamma_{133/3} = \gamma_{233/3} = 0. \end{gathered} \tag{1.4}$$

Throughout this paper, the Greek indices $\alpha, \beta, \gamma, \delta, \epsilon$ have no tensor character, but are merely used to distinguish one tensor (or scalar) from another. We write, for any scalar f, $f_{/\alpha} = (\partial_i f) v^i_{(\alpha)}\,, \partial_i f = \partial f/\partial x^i$.

If g_{ij} has all three Ricci invariants equal, it is a 3-space of constant curvature. The corresponding theorem for this case is known. (See, for example, [6; Theorems 3.3 and 4.3].)

Slebodzinski proved Theorem 1.1 for the particular case where

$$d^*s^2 = [\rho(x^k)]^2\, dt^2 - g_{ij}(x^k)\, dx^i\, dx^j$$

is to be the metric of an Einstein 4-space with ${}^*R_{k\lambda} = 0$, and gave to (1.4) a very simple geometric interpretation. A similar result was given later by Delsarte [1].

At the end of our paper some related results are given without proof.

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