REARRANGEMENT OF CONVERGENT SERIES

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It is well known that every conditionally convergent series of real numbers can be rearranged in such a way that it converges to a preassigned value. However it seems that the permutations of the terms which leave the sum of a convergent series invariant have not yet been investigated. (Two papers with nearly the same heading [1] and [3] have little in common with the present paper.) These permutations will be discussed in this paper from the standpoints of combinatorics, of group theory and of lattice theory. At first the necessary and sufficient conditions will be established that a given permutation leaves invariant either the sum of a given convergent series or the sum of every convergent series (§1). Then it will be shown that the permutations which leave the sum of every convergent series invariant form a semigroup which is not a group; whereas the permutations for which the sum of a given series remains invariant form a set which in general is not even a semigroup (§2). If a permutation Π leaves the sum of a convergent series S invariant, then there exists a binary relation $\Pi \rho S$ between these two entities from which a closure-notion for permutations as well as for convergent series can be derived. The closed sets of permutations (of convergent series) form a complete lattice with inclusion as its lattice-operation. Some properties of these lattices will be discussed (§3). It is remarkable that the notion of "closed set of permutations" obtained by this method does not depend on the fact that the terms and sums of the series run over all the real numbers. Other systems, as specified below, if selected instead of the field of the real numbers, will lead to the same lattice of closed permutations (§4). If one replaces ordinary summation by summability of a higher order, the combinatorial problem becomes more difficult (§5).

1. Given a convergent series $a_1 + a_2 + \cdots$ of real terms and a permutation Π of the terms. For a given index n, let σ_n be the sum of those a_ν, $\nu \leq n$, which by the permutation Π are moved to places right of n, (the "jumping-out" terms) and let τ_n be the sum of those a_μ, $\mu > n$, which are moved to places $\leq n$, (the "jumping-in" terms); then the partial sum $s_n = a_1 + \cdots + a_n$ is changed by Π to $s_n + \tau_n - \sigma_n$ and therefore the necessary condition that Π does not change the sum of the series is

$$\lim_{n\to\infty} \tau_n - \sigma_n = 0. \tag{1}$$

To find out the necessary and sufficient condition that Π does not change the sum of any convergent series S, we subdivide for every n the "jumping-out"

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