INTEGRAL CAYLEY NUMBERS

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The first part of this paper contains a simple proof of the famous "eight square theorem" which expresses the product of any two sums of eight squares as a single sum of eight squares (§2). This proof, like that of Dickson [8; 158–159], employs the system of Cayley numbers, which is somewhat analogous to the field of complex numbers and to the quasi-field of quaternions.

Since the integral complex numbers and integral quaternions were extensively studied by Gauss and Hurwitz (see §3), it is strange that the analogous set of Cayley numbers has been comparatively neglected. There is a short paper by Kirmse [14], but that is marred by a rather serious error, as we shall see in §4. Bruck showed me a simple way to remedy this defect. §§5–12 are concerned with a verification that Kirmse's set of Cayley numbers, as corrected by Bruck, is indeed a set of *integral* elements, according to a precise definition due to Dickson. A geometrical representation is found to be helpful (§6), and it appears that the integral Cayley numbers correspond to the closest packing of spheres in eight dimensions, just as the integral quaternions correspond to the closest packing in four dimensions.

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1. **Cayley numbers: associative and anti-associative triads.** In the notation of Cartan and Schouten [1; 944] the Cayley numbers are sets of eight real numbers

$$[a_0\ ,\ a_1\ ,\ a_2\ ,\ a_3\ ,\ a_4\ ,\ a_5\ ,\ a_6\ ,\ a_7] = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7\ ,$$

which are added like vectors and multiplied according to the rules

$$e_r^2 = -1, \qquad e_{r+1}e_{r+3} = e_{r+2}e_{r+6} = e_{r+4}e_{r+5} = e_r\ ,$$
$$e_{r+3}e_{r+1} = e_{r+6}e_{r+2} = e_{r+5}e_{r+4} = -e_r\ , \qquad e_{r+7} = e_r\ .$$

These rules may be written in the concise form

$$(1.1)\quad e_r^2 = e_1e_2e_4 = e_2e_3e_5 = e_3e_4e_6 = e_4e_5e_7 = e_5e_6e_1 = e_6e_7e_2 = e_7e_1e_3 = -1,$$

provided we interpret "$e_1e_2e_4 = -1$" to mean

$$e_2e_4 = -e_4e_2 = e_1\ , \qquad e_4e_1 = -e_1e_4 = e_2\ , \qquad e_1e_2 = -e_2e_1 = e_4\ ,$$

like the famous relations $i^2 = j^2 = k^2 = ijk = -1$ of Hamilton [12; 339].

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