A CLASS OF ENTIRE FUNCTIONS

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1. **Introduction.** Let us denote by K_A the class of entire functions of exponential type at least A; for all θ, we have $h(\theta) \leq A$, where

$$(1.1) \qquad h(\theta) = \limsup_{r\to\infty} \frac{\log | f(re^{i\theta}) |}{r}.$$

The function $h(\theta)$ describes the rate of growth of the function $f(z)$ in the direction θ. We shall also use the symbol $K(a, c)$ to denote those functions of exponential type which are of type a on the whole real axis, and type c on the whole imaginary axis; then, $h(0) \leq a$, $h(\pi) \leq a$, and $h(\pm\frac{1}{2}\pi) \leq c$, so that for any positive ϵ, $(z) = O(1) \exp (a | x | + c | y | + \epsilon | z |)$ as $r \to \infty$.

One of the central problems of the study of the class K_A is that of inferring properties of $f(z)$ from the sequence of values $\{f(n)\}$. (For a discussion of this and related problems as well as an extensive bibliography, see [1].) Suppose we are given an arbitrary sequence of complex numbers $\{w_n\}$, can we find a function of K_A such that $f(n) = w_n$ and is it unique? First, an obvious necessary condition for the existence of such a function is that the growth of the sequence $\{w_n\}$ be of finite type, i.e.

$$(1.2) \qquad \limsup | w_n |^{1/n} < \infty .$$

This condition is also sufficient. In fact, if (1.2) holds, there is a function $f(z)$ of $K(a, \pi)$ for which $f(n) = w_n$. We need only choose $f(z)$ as $e^{\alpha z}g(z)$, where

$$g(z) = \frac{\sin \pi z}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n w_n e^{-\alpha n}}{z - n}.$$

If $\alpha > \limsup (\log | w_n |)/n$, then this series converges for all z, and $g(z)$ belongs to $K(0, \pi)$.

Thus, the class $K(a, \pi)$ is universal in the sense that in it all possible sequences of values $\{f(n)\}$ are achieved. We cannot have uniqueness, since to $f(z)$ we can add $h(z) \sin \pi z$ where $h(z)$ belongs to K_0 without altering the class. The question now arises: can we give necessary or sufficient conditions on the sequence w_n for there to exist a function $f(z)$ taking these values at the integers, and belonging to the class $K(a, c)$ for some a and some $c < \pi$? A well-known theorem of Carlson assures us that if there is such a function, it is then unique.

We shall formulate one such necssary and sufficient condition, and then make a number of applications of it. In §4, we shall discuss the effects of oscillation in sign of the real parts of the numbers $f(n)$; in §6 and §7, we study the characterization of integral-valued entire functions and the more special problem of

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