THE KERNEL FUNCTION OF AN ORTHONORMAL SYSTEM

By Menahem Schiffer

1. Let B be a finite domain in the complex z-plane bounded by a finite number of smooth curves $C_1, C_2, \cdots, C_n$. A set of functions $\varphi_\nu(z)$ ($\nu = 1, 2, \cdots$), analytic in B and satisfying the condition

$$(1) \qquad \iint_B \varphi_\mu(z)\overline{\varphi_\nu(z)}\, d\omega_z = \delta_{\mu\nu}, \qquad z = x + iy, \qquad d\omega_z = dx\, dy,$$

is called orthonormal with respect to B.

The family of all functions $f(z)$, analytic in B and with $\iint_B |f(z)|^2\, d\omega_z < \infty$, will be denoted by $L^2(B)$. If every $f(z) \subset L^2(B)$ may be expanded into a series

$$(2) \qquad f(z) = \sum_{\nu=1}^{\infty} c_\nu \varphi_\nu(z), \qquad c_\nu = \iint_B f(\zeta)\overline{\varphi_\nu(\zeta)}\, d\omega_\zeta$$

which converges uniformly in every interior sub-domain of B, the set $\varphi_\nu(z)$ is called a closed system.

Bergman [1, 2] proved for every domain B the existence of closed orthonormal systems and indicated a method for their construction. For every given domain B there exist an infinity of closed orthonormal sets. Bergman further defines

$$(3) \qquad K(z; \bar{\zeta}) = \sum_{\nu=1}^{\infty} \varphi_\nu(z)\overline{\varphi_\nu(\zeta)}$$

to be the kernel function of B. He shows that the infinite series in (3) converges uniformly if z and ζ are restricted to an interior partial domain of B and that, therefore, $K(z; \bar{\zeta})$ is an analytic function of z and $\bar{\zeta}$. $K(z; \bar{\zeta})$ is independent of the particular closed orthonormal set $\varphi_\nu(z)$, and is uniquely determined by the domain B. The kernel function plays an important role in the theory of conformal representation [2, 3] and is of considerable practical importance, especially as its numerical computation may be performed with ease.

There arises, however, the question of how $K(z; \bar{\zeta})$ is related to the other domain functions which appear in the theory of conformal representation, in particular to Green's function. In this paper, this relation will be established and thereby a twofold purpose will be attained. On the one hand, all theoretical results obtained with respect to Green's function become at once applicable to the kernel function; on the other hand, convenient methods for computing the kernel function are thus made available for practical applications of Green's function. It will be shown below that certain results concerning Green's function

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