ELLIPTIC GEOMETRY, CONFORMAL MAPS, AND ORTHOGONAL MATRICES

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1. **Introduction.** The well-known connection of two-dimensional elliptic geometry with ternary orthogonal matrices depends upon the derivation of this branch of non-euclidean geometry from geometry on the surface of a sphere by identification of diametrically opposite points. The measurement of distances on the sphere by stretches of great circles possibly accounts for the usual emphasis that is laid upon the formulae of spherical trigonometry as representing the most suitable algebraic apparatus for the study of elliptic geometry. In the case of two-dimensional hyperbolic geometry, on the other hand, the interior of a unit circle equipped with a Lobatchevsky metric is generally used as a model. The ease with which this model can be manipulated rests upon the coincidence of the conic upon which the metric is based with the boundary of the space—the unit circle itself. In the following, the interior of the unit circle in the complex z-plane is employed as a model for the elliptic plane and a system of coordinates introduced which, while neither isothermal nor orthogonal, makes the connection with ternary orthogonal matrices immediate. The system is particularly suitable for practical applications such as gyrostatics, because it is easily constructed and plotting of curves may be carried out directly without necessitating conversion to the (here superimposed) euclidean system of the z-plane. The one-sided, or non-orientable, character of the elliptic plane also becomes evident. When in the following it is found desirable to maintain this coordinate system intact under transformations, the points or other configurations considered will be assumed to move in the fixed system—a well-known device that is frequently found to be expedient.

The integration of the system of linear differential equations, expressed as follows in matrix form

$$(1) \qquad \frac{dT}{dt} = T \cdot \Omega, \qquad (\Omega = -\Omega^*),$$

where Ω is a given ternary skew symmetric matrix with real elements and the asterisk indicates the transposed matrix, will afford an opportunity for applying this geometry. That this so-called selfadjoint system is integrated by an orthogonal matrix T for which $T \cdot T^* = 1$ is almost trivial, and can be shown formally for any finite number of dimensions as follows:

$$\frac{dT}{dt} \cdot T^* = T \cdot \Omega \cdot T^* \text{ and } T \cdot \frac{dT^*}{dt} = (T \cdot \Omega \cdot T^*)^* = -T \cdot \Omega \cdot T^*$$

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