THE GROWTH OF ANALYTIC FUNCTIONS

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1. Let $\varphi(z)$ be analytic in the half-plane $x \geq 0$ and of finite exponential type there; that is, $|\varphi(z)| \leq Ae^{B|z|}$ for some constants A and B. The problem considered in this paper is that of when

$$(1.1) \qquad \limsup_{x\to\infty} x^{-1} \log |\varphi(x)| = \limsup_{x\to\infty} \lambda_n^{-1} \log |\varphi(\lambda_n)|.$$

We consider only real sequences $\{\lambda_n\}$, although our results can be extended to complex sequences. We suppose that $|\lambda_{n+1} - \lambda_n| \geq d > 0$ and that $\{\lambda_n\}$ has a density. That is, $\lim_{n\to\infty} n/\lambda_n = D$, $0 \leq D < \infty$, or, equivalently, $\lim_{t\to\infty}\Lambda(t)/t = D$, where $\Lambda(t)$ denotes the number of λ_n not exceeding t.

It was shown by V. Bernstein [1; 230] that (1.1) is true if

$$(1.2) \qquad \limsup_{|y|\to\infty} |y|^{-1} \log |\varphi(iy)| = \pi L$$

with $D > L$. Bernstein's proofs involved deep results from the theory of Dirichlet series. Levinson [5], [6; Chapter 7] and Pfluger [8] gave simpler, essentially equivalent, proofs by ordinary function theory. Levinson [6; Chapter 7] also extended the result to cases where D may be equal to L, but (1.2) is strengthened, using theorems on Fourier transforms. Here we shall show that the idea of Levinson's and Pfluger's proofs of Bernstein's theorem can be used to establish both Levinson's sharper theorems and still more general ones, and even to relax the hypothesis that $\varphi(z)$ is of finite type. Precise statements are given below. While we deal only with functions in a half-plane, our results could easily be extended to any angle, corresponding to Bernstein's original theorem.

Our proofs depend on the following theorem of Phragmén-Lindelöf type, which can also be used in other problems.

If $f(z)$ is of exponential type in $x \geq 0$ and is bounded on the curve $\cos\theta = \delta(r)$, *where* $0 \leq \delta(r) < \frac{1}{2}$ *and* $\int^\infty r^{-1}\delta(r)dr$ *diverges, then $f(z)$ is bounded in $x \geq 0$.*

A more general statement (Theorem 1) is given in §2; the requirement that $f(z)$ is of exponential type is unnecessarily restrictive. The proof depends on a mapping theorem of Ahlfors [7; 92]. (Theorem 1 is also implicit in the work of Fuchs [3].)

Theorem 2. *Let $\varphi(z)$ be analytic and of finite exponential type in $x \geq 0$ and satisfy*

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