# A SELF-ADJOINT DIFFERENTIAL SYSTEM OF EVEN ORDER

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1. **Introduction.** This paper is concerned with an extension of the results of Reid [5, §11] to a self-adjoint boundary value problem

$$(1.1)\qquad F(u) = \lambda G(u), \quad U_\sigma(u;\lambda) \equiv U_\sigma^0(u) + \lambda U_\sigma^1(u) = 0 \qquad (\sigma = 1, \cdots, 2n),$$

where $F(u)$ and $G(u)$ are differential operators of the form

$$F(u) = \sum_{\nu=0}^{n} (f_\nu(x)u^{(\nu)})^{(\nu)}, \qquad G(u) = \sum_{\mu=0}^{n-1} (g_\mu(x)u^{(\mu)})^{(\mu)},$$

$f_n(x) \neq 0$, $f_\nu(x)$ ($\nu = 0, 1, \cdots, n$) are real functions of class $C^{(\nu)}$ on the finite interval $a \leq x \leq b$ and $g_\mu(x)$ ($\mu = 0, 1, \cdots, n-1$) real functions of class $C^{(\mu)}$ on $ab$, while for arbitrary values of $\lambda$ the $U_\sigma(u;\lambda)$ are independent linear forms in the end values of $u, u', \cdots, u^{(2n-1)}$ at $x = a$ and $x = b$ with real coefficients for which $U_\sigma^1(u)$ involves only the end values of $u, u', \cdots, u^{(n-1)}$.

Reid has shown that systems (1.1) with $G(u) = g_0(x)u$ and boundary conditions independent of $\lambda$ are of a type associated with a problem of Bolza in the calculus of variations. In §2 it is shown that systems (1.1) are equivalent to the Euler-Lagrange differential equations and transversality conditions for minimizing a quadratic functional (see Bobonis [1, §10])

$$(1.2)\qquad J_2(\eta) \equiv 2\mathfrak{Q}(\eta(a), \eta(b)) + \int_a^b 2\omega(x, \eta, \eta')\,dx,$$

where $\mathfrak{Q}$ and $\omega$ are quadratic forms in the variables $\eta_i(a)$, $\eta_i(b)$ ($i = 1, \cdots, n$) and $\eta_i$, $\eta_i'$, respectively, in a class of arcs $\eta_i(x)$ ($a \leq x \leq b$) which satisfy a set of ordinary linear differential equations of the first order

$$\Phi_\alpha(x, \eta, \eta') \equiv \Phi_{\alpha\eta'_j}(x)\eta_j' + \Phi_{\alpha\eta_j}(x)\eta_j = 0$$

$$(\alpha = 1, \cdots, m < n; j = 1, \cdots, n),$$

the linear homogeneous end conditions

$$\Psi_\gamma(\eta(a), \eta(b)) \equiv \Psi_{\gamma j}^1\, \eta_j(a) + \Psi_{\gamma j}^2\, \eta_j(b) = 0 \qquad (\gamma = 1, \cdots, p \leq 2n),$$

and an isoperimetric condition requiring the quadratic expression

$$(1.3)\qquad K_2(\eta) \equiv 2\mathfrak{G}(\eta(a), \eta(b)) + \int_a^b \eta_i K_{ij}(x)\eta_j\,dx,$$

where $\mathfrak{G}$ is a quadratic form in $\eta_i(a)$, $\eta_i(b)$, to be equal to a given constant value.

Hölder [2] has shown that systems (1.1) can be regarded as canonical systems

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