LIMITS FOR THE CHARACTERISTIC ROOTS OF A MATRIX

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Let $A = (a_{\kappa\lambda})$ be an arbitrary n-square matrix, and g the maximum of the absolute values of the elements $a_{\kappa\lambda}$. It was proved by A. Hirsch [6] that each characteristic root ω_ν of A satisfies the inequality

$$(1) \qquad | \omega_\nu | \leq ng.$$

Another proof of this inequality was given by Bromwich [1]. I. Schur [11] proved the sharper result

$$(2) \qquad \sum_{\nu=1}^{n} | \omega_\nu |^2 \leq \sum_{\kappa,\lambda=1}^{n} | a_{\kappa\lambda} |^2.$$

It follows from (2) that equality holds in (1) only if

$$(3) \qquad a_{\kappa\lambda} = ae^{i(\varphi+\varphi_\kappa-\varphi_\lambda)},$$

where $\varphi, \varphi_1, \varphi_2, \cdots, \varphi_n$ are arbitrary real numbers.

Denote the sum of the absolute values of the elements in the κ-th row by R_κ, the sum of the absolute values of the elements in the λ-th column by T_λ, and the maxima of the R_κ and of the T_λ by R and T, respectively. Frobenius [5] proved that

$$(4) \qquad | \omega_\nu | \leq \min (R, T)$$

if all the elements $a_{\kappa\lambda}$ are positive. It was shown by E. T. Browne [2] that

$$(5) \qquad | \omega_\nu | \leq \tfrac{1}{2}(R + T)$$

if the elements $a_{\kappa\lambda}$ are arbitrary real or complex numbers.

Let S_ρ be the sum of the absolute values of the elements in the ρ-th row and the absolute values of the elements in the ρ-th column, and S be the maximum of the S_ρ. W. V. Parker [8] obtained

$$(6) \qquad | \omega_\nu | \leq \tfrac{1}{2}S$$

which is in general better than (5). Since the geometric mean is not greater than the arithmetic mean, the following result of A. B. Farnell [6] is sharper than (5)

$$(7) \qquad | \omega_\nu | \leq (RT)^{\frac{1}{2}}.$$

Moreover, Farnell proved that

$$(8) \qquad | \omega_\nu | \leq \left[\sum_{r=1}^{n} \left(U_r V_r\right)^{\frac{1}{2}} \right]^{\frac{1}{2}},$$

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