AN EXTENSION OF CARLSON'S THEOREM

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Let A be a subset of the set I of all positive integers; we will denote its complement in I by A'. Let $\{\gamma\}_A$ be the class of all real numbers γ such that there exists a function $f(z)$ in $K^*(a, \gamma)$ which vanishes in A but is not identically zero. $K^*(a, c)$ is the class of functions which are regular in $R\{z\} \geq 0$, with $f(z) = O(1)\ e^{a|x|+c|y|+\epsilon|z|}$ for any $\epsilon > 0$. $\{\gamma\}_A$ is not void, but always contains the number π, since $\sin \pi z$ vanishes at I, but not identically. Finally, we denote the greatest lower bound of the set $\{\gamma\}_A$ by $\gamma(A)$. The following is an obvious consequence of these definitions.

Theorem 1. *If $f(z) \in K^*(a, c)$, $c < \gamma(A)$, and if $f(z)$ vanishes in A, then $f(z) \equiv 0$.*

It is also clear that if $A \subset B$, then $\{\gamma\}_A \supset \{\gamma\}_B$ and $\gamma(A) \leq \gamma(B)$. A theorem of Carlson asserts that $\gamma(I) = \pi$ [1]. More generally, if A has a density, $D(A)$, then $\gamma(A) = \pi D(A)$. We shall be concerned with extensions of this to the case where A fails to have a density in the usual sense.

We shall denote the Pólya *maximum density* of A by $\overline{D}_1(A)$

$$\overline{D}_1(A) = \lim_{\theta \to 1-0} \overline{\lim_{x \to \infty}} \frac{A(x) - A(\theta x)}{x - \theta x}, \tag{1}$$

where $A(x)$ is the number of points of A in the interval $(0, x)$; $\underline{D}_1(A)$, the *minimum density* of A is defined similarly, replacing $\overline{\lim}$ by $\underline{\lim}$. The *upper* and *lower* densities of A are defined as $\overline{D}(A) = \overline{\lim}\, A(x)/x$, and $\underline{D}(A) = \underline{\lim}\, A(x)/x$. If $\overline{D}(A) = \underline{D}(A)$, then A belongs to the class $\mathfrak{D}$ of sets having a density. It is easily shown that $\underline{D}_1(A) \leq \underline{D}(A) \leq \overline{D}(A) \leq \overline{D}_1(A)$. The theorem of Carlson may now be stated in a modified form.

Theorem 2. $\pi \underline{D}_1(A) \leq \gamma(A) \leq \pi \overline{D}_1(A)$.

For, if $\overline{D}_1(A) = \beta$, then by a property of maximum density [3; 562], there is a set B containing A of density β and $\gamma(A) \leq \gamma(B) = \pi D(B) = \pi\beta = \pi \overline{D}_1(A)$. The argument for $\underline{D}(A)$ proceeds similarly, since there is also a set C contained in A of density $\underline{D}_1(A)$.

Theorem 3. $\gamma(A) + \gamma(A') \geq \pi$.

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