# THE PROBLEM OF THE ROOKS AND ITS APPLICATIONS

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1. **Introduction.** The classification of permutations by ascending runs was first studied by MacMahon in [8; I, §IV, Chapters IV, V] and [9]. Applications of his results to significance tests in statistics have been made recently by Moore and Wallis [11] and Mann [10].

It does not seem to be generally known that the numbers arising in this study have occurred several times in other contexts. Rather extensive references to older literature (going back to Laplace) are given by v. Schrutka in [13], who however was unaware of MacMahon's work. More recently the numbers have appeared in papers [2], [3] by Dwyer, who calls them "cumulative numbers", and in papers [14], [15] by Toscano on summation of series.

In this paper we shall endeavor to show that these results may be unified and generalized by the study of a chessboard recreation: in how many ways can a given number of rooks be placed on a chessboard so that no two attack each other? We shall not identify solutions which are equivalent under the symmetry group of the board (cf. [7; 240–247] for a study of this version of the problem), so that for us the problem on a rectangular board is trivial. The board of primary interest is the trapezoid, and in particular the triangle. A triangle mutilated so as to become an irregular ramp-staircase is shown in §7 to be associated with what MacMahon called "Simon Newcomb's problem", for which we are able to give a compact solution. In §8 a board composed of two triangles is studied; the application to bi-triangular permutations there described arose in connection with war work. Finally Ahrens's problem of the bishops [1; 140–151] is solved in §9 by an application of the results of §7.

As noted in §3, this study is closely connected with the card-matching problem treated by symbolic methods in [5] and [6]. While there is no fundamental difference between the two methods, we believe that in the present context the terminology of the chessboard is conducive to clarity and simplicity of proof.

2. **Chessboards.** A general chessboard $A$ may be formally defined as a set of points $(x, y)$ with integral coordinates. Let $u_k(A)$ denote the number of ways of choosing $k$ points in $A$ so that no two are in the same row or column, i.e., no two have the same $x$- or $y$-coordinate. In the language of the chessboard, $u_k(A)$ is the number of ways of placing $k$ non-attacking rooks on $A$. We also use the generating function $U(A) = \sum_{k=0}^{\infty} u_k(A)x^k$.

Two boards $A$ and $B$ may be said to be *disjoint*, if no point of $A$ is in the same row or column as a point of $B$. If the board $C$ can be decomposed into two disjoint sub-boards $A$ and $B$, then evidently $U(C) = U(A)U(B)$.

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