MAXIMAL IDEMPOTENT SETS IN A RING WITH UNIT

BY ALFRED L. FOSTER

1. **Introduction.** The present paper is concerned with (1) a certain extension to general rings (with unit) of results on idempotent ring subsystems previously obtained for commutative rings [1] (see also Theorems A and B below), and with (2) applications of this extension to interesting classes of full matrix rings in which the matrix elements variously belong to (a) a field, (b) certain domains of integrity, including the ring of whole numbers, (c) certain non-domains of integrity, including $W/(m)$, the ring of whole number residues mod m.

The background of the paper, as in [1], is that of the general ring duality theory initiated in [4] and extended in [1], [3], and [2],—a duality theory for rings which embraces the familiar Boolean duality. To insure proper orientation we shall very briefly recall a few of the basic notions of this theory. (In [4] the ring duality theory was presented and applied only to commutative rings (with unit). The basic ideas of [4], however, including the portions here reviewed, do not require this commutativity, as shown in the latter part of [1]; in fact it is there shown that the duality theory (or better theories) is properly a part of a general transformation theory, and applies to algebras of the most general kind.)

Let $R = (R, +, \times)$ be a ring with unit. The concepts and theorems of R may be arranged in *dual pairs*. In particular 0 and 1 are dual elements, and each of the pairs of operations $+, \oplus$; $-, \ominus$; $\times, \otimes$; $*$; consists of dual operations (the last being self-dual), where

$$a \oplus b = a + b - 1 \qquad \text{(dual addition)}$$

$$a \ominus b = a - b + 1 \qquad \text{(dual subtraction)}$$

$$a \otimes b = a + b - ab \qquad \text{(dual product, also called } \textit{Logical ring sum}\text{)}$$

$$a^* = 1 - a \qquad (\textit{ring complement}).$$

More generally, if $\varphi(x, y, \cdots)$ is any operation (function) of one or more R-variables $x, y, \cdots$ mapping R onto (all or part of) itself, the dual or *transform* (see [1]) of φ is given by

$$dl\ \varphi(x, y, \cdots) = \varphi^*(x^*, y^*, \cdots).$$

Specialized to the above set of operations, one has the following restricted form of the

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