A THEOREM OF M. BAUER

By Alfred Brauer

Bauer [1] has proved the following theorem.

Let $f(x)$ be a polynomial with integral rational coefficients and at least one real root of odd multiplicity. If all the prime divisors of $f(x)$, with the exception of a finite number, have the form $kz \pm 1$ where $k > 2$ is an integer, then $f(x)$ has an infinite number of prime divisors which do not have the form $kz + 1$.

Bauer's proof is also presented by E. Landau [2; 440–441]. I. Schur [3] remarked that the theorem holds for every integral polynomial with at least one real root.

In the following, I shall give another simple proof of Bauer's theorem which will show that the theorem is almost trivial for polynomials of odd degree. I shall prove the following generalization of Bauer's result.

Theorem 1. *Let $f(x)$ be a polynomial with integral rational coefficients which has at least one real root. Let $\mathfrak{G}(k)$ be the group of the residue classes relatively prime to k, and $\mathfrak{H}$ a subgroup which does not contain the class of numbers congruent to* $-1 \pmod{k}$. *Then $f(x)$ contains infinitely many prime divisors which do not belong to the classes of $\mathfrak{H}$.*

Since the quadratic residues form a subgroup $\mathfrak{H}$ of $\mathfrak{G}(k)$, and since -1 is a quadratic non-residue for the primes q of form $4n + 3$, it follows from Theorem 1 at once

Theorem 2. *If q is a prime of form $4n + 3$ and $f(x)$ a polynomial with a real root, then $f(x)$ contains an infinite number of prime divisors which are quadratic non-residues* mod q.

For every k, polynomials exist of which all the prime divisors, except a finite number, have the form $kz + 1$. It is unknown whether polynomials exist of which all the prime divisors, except a finite number, belong to the same residue class $kz + l$ with $l \neq 1$. Here the following result is obtained.

Theorem 3. *Let $f(x)$ be an integral polynomial with a real root. Let k be an integer of one of the following forms: 2^α, $2^\beta P$, $2Q$, or Q where P and Q are Fermat primes $2^{2^\gamma} + 1$ or products of different Fermat primes, Q divisible by 3, and $\beta \geq 2$. Assume that all the prime divisors of $f(x)$ with the exception of a finite number belong to the same residue class $kz + l$. Then* $l \equiv -1 \pmod{k}$.

Proof of Theorem 1. It is sufficient to assume that $f(x)$ is irreducible in the field of rational numbers. Otherwise we consider a factor of $f(x)$ which has a

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