REMARKS ON THE CARLITZ ψ-FUNCTIONS

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1. Introduction. Let $GF(p^n)$ denote a finite field of order p^n, $GF[p^n, x]$ the ring of polynomials in a single indeterminant x over $GF(p^n)$ and $GF(p^n, x)$ the corresponding field of rational functions in x. L. Carlitz [1] has defined certain functions closely related to the arithmetic of $GF[p^n, x]$. Of particular interest is the function

$$\psi(t) = \psi(t; n) = t \prod \left(1 - \frac{t}{E\xi}\right), \tag{1}$$

the product extending over all non-zero elements of $GF[p^n, x]$, and with ξ in a suitable extension of $GF(p^n, x)$. (For convergence and related questions see §4 below.) It was proved that

$$\psi(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{F_j} t^{p^{nj}}, \tag{2}$$

where

$$F_j = [j][j-1]^{p^n} \cdots [1]^{p^{n(j-1)}}; \qquad [j] = x^{p^{nj}} - x; \qquad F_0 = 1. \tag{3}$$

The purpose of the present note is to study some of the properties of the function $\psi(t)$ from the standpoint of general theorems on power series in algebraically closed fields that are complete with respect to a non-archimedian valuation, and also to give certain theorems that will be needed in subsequent papers. In particular, we shall start with the infinite series (2) as definition of $\psi(t)$ and deduce (1).

2. Power series. Let $\mathfrak{M}$ be an algebraically closed field with a non-archimedian valuation v (real-valued, in the sense of Kürshák [2]) with respect to which $\mathfrak{M}$ is complete (or perfect, see [7]). For $a \in \mathfrak{M}$ we shall refer to $v(a)$ as the modulus of a. Then $m(a, b) = v(a - b)$ defines a metric for the set $\mathfrak{M}$, and it is with respect to the corresponding metric topology that $\mathfrak{M}$ is assumed complete. Since $\mathfrak{M}$ is algebraically closed v cannot be a discrete valuation [3; 24] and the values of v must be dense in the set of positive real numbers [7]. Since v is non-archimedian $v(a) > v(b)$ implies that $v(a + b) = v(a)$, and it follows readily that $\mathfrak{M}$ is not locally compact. $\mathfrak{M}$ is totally disconnected.

A well-known necessary and sufficient condition for the convergence of an infinite series $\sum a_n$ or an infinite product $\prod (1 - a_n)$ is that $\lim_{n\to\infty} v(a_n) = 0$. The convergence is always unconditional; that is, arbitrary rearrangement of

Received December 3, 1945; these results were obtained while the author was a National Research Fellow, 1942–1943, at the Institute for Advanced Study.