THE TOPOLOGY OF PSEUDO-HARMONIC FUNCTIONS

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1. **Introduction.** The relations between the zeros, poles, and branch points of a meromorphic function $f(z)$ defined in a *limited open region* G bounded by ν Jordan curves can be deduced from the relations between the logarithmic poles and critical points of the harmonic function

$$u(x, y) = \log |f(z)| \qquad (z = x + iy).$$

If this study is to be made by topological methods, as we propose, it is natural to replace the class of harmonic functions by the class of *pseudo-harmonic functions* defined as follows.

Let $u(x, y)$ be a function which is harmonic and not identically constant in the neighborhood N of a point (x_0, y_0). Let the points of N be subjected to an arbitrary homeomorphism T in which N corresponds to another neighborhood N' of (x_0, y_0) and the point (x, y) on N corresponds to a point (x', y') on N'. It will be convenient to suppose that (x_0, y_0) corresponds to itself. Under T set

$$u(x, y) = U(x', y'). \tag{1.1}$$

The function $U(x', y')$ will be termed *pseudo-harmonic* on N'.

This definition will be extended to the case where $u(x, y)$ has a logarithmic pole at (x_0, y_0). In this case

$$u(x, y) = k \log r + \omega(x, y) \qquad (k \neq 0), \tag{1.2}$$

where r is the distance from (x, y) to (x_0, y_0), k is a real constant, and $\omega(x, y)$ is harmonic in a neighborhood of (x_0, y_0). If $U(x', y')$ is defined by (1.1), subject to T as previously, $U(x', y')$ will be said to be *pseudo-harmonic* on N' *except for a logarithmic pole* at (x_0, y_0).

Let $\bar{G}$ denote the closure of G and βG the boundary of G. We suppose $U(x, y)$ defined on $\bar{G}$ except for poles.

We shall admit functions $U(x, y)$ which are pseudo-harmonic, except for poles, in some neighborhood of every point of G, and which are continuous at every point of $\beta\bar{G}$.

It is worthwhile to mention some of the advantages and disadvantages of studying pseudo-harmonic functions instead of harmonic functions. Some of the advantages follow:

(1) The essentially topological relations governing the number and distribution of the logarithmic poles and critical points are brought out. It is seen that the concept of critical point, topologically defined, is *relative* both to the function U and to $\bar{G}$. ("Relative" here is used in a sense that will become clear in §3. The

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