THE EXISTENCE OF SOLUTIONS TO LAGRANGE PROBLEMS FOR MULTIPLE INTEGRALS

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1. **Introduction.** We consider the problem of minimizing the Lebesgue integral

$$\int_G f(x_1, \cdots, x_N, z_1, \cdots, z_m, D_1 z_1, \cdots, D_{x_N} z_m)\, dx_1 \cdots dx_N$$

over a region G in E_N among a suitable class of admissible functions z on G to E_m (where $z_1, \cdots, z_m$ are such functions that $z(x) = (z_1(x), \cdots, z_m(x))$ for every x in G) which have boundary values in a given family Γ and which satisfy a set of quasilinear differential equations

$$\theta_i(x_1, \cdots, x_N, z_1, \cdots, z_m, D_{x_1} z_1, \cdots, D_{x_N} z_m) = 0 \qquad (i = 1, \cdots, r)$$

almost everywhere on G. The class $\mathfrak{P}_1$ of potential functions integrable together with their generalized derivatives on a region G which were introduced by G. C. Evans [2] and extensively studied by C. B. Morrey, Jr., [1] and [4], form a class of functions of which our admissible functions are a subclass.

2. **The class $\mathfrak{P}_\alpha (\alpha \geq 1)$.**

2.1. Definition. A function on a region G to E_m is *of class* $\mathfrak{P}$ on G if and only if each of its components is a function on G to E_1 of class $\mathfrak{P}$ on G [4; Definition 1].

2.2. Definition. If z on G to E_m is of class $\mathfrak{P}$ on G with components $z_1, \cdots, z_m$ and generalized derivatives [4; Definition 5] $D_{x_1} z_1, \cdots, D_{x_N} z_m$, $\alpha \geq 1$, then we define

$$D_\alpha(z, G) = \int_G [\sum_{j=1}^{m} \sum_{i=1}^{N} (D_{x_i} z_j)^2]^{\frac{1}{2}\alpha}\, dx,$$

$$|| z ||^\alpha = D_\alpha(z, G) + \int_G [\sum_{j=1}^{m} z_j^2]^{\frac{1}{2}\alpha}\, dx.$$

If $|| z ||$ is finite, z is said to be *of class* $\mathfrak{P}_\alpha$ on G. We read $|| z ||$ as *the norm of z in $\mathfrak{P}_\alpha$ on G.*

2.3. Theorem. *The space $\mathfrak{P}_\alpha$ ($\alpha \geq 1$) of classes of equivalent functions on G to E_1 of class $\mathfrak{P}_\alpha$ on G is a Banach space.*

Received January 23, 1945; in revised form June 27, 1945.