INTEGRAL CRITERIA FOR SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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1. **Introduction.** The present paper is concerned with integral criteria for a function to be equal a. e. (almost everywhere) to a solution of a linear differential equation. The criteria involve, in particular, the forward differences $\Delta_h f(x) = f(x+h) - f(x)$, $\Delta_h^p f(x) = \Delta_h[\Delta_h^{p-1} f(x)]$, $p = 2, 3, \cdots$, of $f(x)$ corresponding to a positive value of h. We shall denote by $o(h^n)$ a generic quantity which is such that $o(h^n)/h^n$ approaches zero as h tends to zero through positive values. The following result is established in §3.

Theorem I. *If $f(x)$ is integrable on an interval $(a, b+d)$, and*

$$\int_a^b | \Delta_h^n f(x) | \, dx = o(h^n), \tag{1.1}$$

then there exists a polynomial $P_{n-1}(x)$ of degree at most $n-1$ such that $f(x) = P_{n-1}(x)$ a. e. on (a, b).

This result is a special case of the following theorem proved in §4.

Theorem II. *Suppose that $a_i(x)$, $i = 0, 1, \cdots, n$, is of class $C^{(i)}$, and $a_n(x) \neq 0$, on $(a, b+d)$, while $g(x)$ and $f(x)$ are integrable on this interval. If*

$$\int_a^b \left| \sum_{i=1}^{n} a_i(x) h^{n-i} \Delta_h^i f(x) + h^n[a_0(x) f(x) + g(x)] \right| dx = o(h^n), \tag{1.2}$$

then there exists a solution $u(x)$ of the differential equation

$$\sum_{i=0}^{n} a_i(x) u^{(i)} + g(x) = 0 \tag{1.3}$$

such that $f(x) = u(x)$ a. e. on (a, b).

The case $n = 1$ of Theorem I is due to Titchmarsh [5]; so far as the author is aware, however, the above indicated generalizations have received no previous attention. The proof of Theorem I given in §3 utilizes the du Bois-Reymond form of the fundamental lemma of the calculus of variations, together with properties of integral means of functions; the pertinent properties of such integral means are listed in §2. Correspondingly, the proof of Theorem II in §4 employs a generalization of the fundamental lemma of the calculus of variations; a discussion of this generalization, which is a slight extension of a result of Hobson [2], is presented in §5. Section 6 is devoted to further remarks on

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