# STAR CENTER POINTS OF MULTIVALENT FUNCTIONS

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1. **Introduction.** Let the function

$$f(z) = a_1z + a_2z^2 + \cdots + a_nz^n + \cdots$$

be regular for $|z| < R$. The function will be called multivalent of order $p$ ($p = 1, 2, \cdots$) relative to the circle $|z| = R$ if no value of the function is taken on for more than $p$ values of $z$ within the circle $|z| = R$ and if at least one value of the function is taken on exactly $p$ times. When $p$ is one the function will be called univalent.

Much has been written on the class of univalent functions which are star-forming with respect to the origin [4]. For such functions the ray from the origin to the point $f(z)$ turns continuously in the anti-clockwise direction as $z$ traverses the circle $|z| = r < R$ in the same direction. For such functions it is well known that for each $r$, $0 \leq |z| = r < R$

$$\mathcal{R}[zf'(z)/f(z)] > 0.$$

On the other hand, very little seems to have been published on univalent functions which are star-forming with respect to points other than the origin [1], nor on multivalently star-forming functions [2].

We shall call $\zeta$, a point within the circle $|z| = r \leq R$, a star center point for the function $f(z)$ relative to the circle $|z| = r \leq R$ whenever the radius vector joining $f(\zeta)$ to $f(z)$, $|z| = r$, $f(\zeta) \neq f(z)$, turns about the center $f(\zeta)$ continuously in the anti-clockwise direction as $z$ in a similar way traverses the circle $|z| = r$. If $f(z)$ is multivalent of order $p$ this vector in the $f(z)$-plane will make $p$ revolutions as $z$ makes one revolution in the $z$-plane. For a large class of multivalent functions $f(z)$ there will exist a non-empty set $\mathfrak{M}_r(f)$ of these star center points. For example, the function $z + 2z^2$, multivalent of order 2 in the unit circle, has for its set $\mathfrak{M}_1$ all the points $z$ of the unit circle which map into the interior of the inner loop of that contour into which $|z| = 1$ is mapped. In particular it includes the points on the real axis $-1 < x < 2^{\frac{1}{2}} - 1$.

It is the purpose of this paper to obtain some properties satisfied by the function $f(z)$ at these points $\zeta$. The results will consist largely of inequalities existing between the points $\zeta$, the coefficients $a_n$ of the power series for $f(z)$ and the value of $f(\zeta)$. In particular, the well-known inequalities $|a_2| \leq 2$, $|a_3| \leq 3$ for univalent functions are obtained in a more precise and sharpened form in terms of $\zeta$, $f(\zeta)$ for univalent functions having $\zeta \neq 0$ as a star center point. Analogous inequalities are also found for star-forming functions multivalent of

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