INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED WITH A LINEAR CONGRUENCE

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1. **Introduction.** In a paper [2] published some years ago, Vogt studied the space transformations associated with a linear congruence, using synthetic methods. Recently the present author [1] gave an analytical discussion of the non-involutorial transformations associated with these configurations, pointing out some properties overlooked by Vogt's study. The involutorial transformations involved are dismissed by Vogt, in the paper mentioned above, with little more than mere mention. In the present paper these transformations are more fully discussed, chiefly by the use of analysis. A number of interesting properties are thus disclosed, among these being the existence of a rather large number of parasitic lines.

Given the directrices r and s of the congruence and a pencil of surfaces

$$| F_{m+n+2} |: r^m s^n g_{2mn+4m+4n+4} \, .$$

Through a generic point $P(y)$ there passes a single F of $| F |$. The unique line t of the congruence through $P(y)$ meets F a second time in one residual point $Q(x)$, the image of $P(y)$ under the transformation thus defined. The residual base curve of $| F |$ has been denoted by g. As indicated above, the surfaces of the pencil are of order $m + n + 2$, r and s lying m times and n times, respectively, on each, while g is of order $2mn + 4m + 4n + 4$. It will be shown that r, s, and g are fundamental curves of the involution.

2. **Equations of the transformation.** Let us take the equations of the directrices r and s, respectively, as

$$x_1 = x_2 = 0, \qquad x_3 = x_4 = 0, \tag{1}$$

and the pencils of surfaces $| F |$ as

$$| F_{m+n+2} | \equiv U - \lambda U' = 0, \tag{2}$$

where

$$\begin{aligned} U_{m+n+2} &= \sum_{j,k=0}^{m,n} u_j v_k (c_{jk} x), & U'_{m+n+2} &= \sum_{j,k=0}^{m,n} u'_j v'_k (c'_{jk} x), \\ u_j &= \sum_{i=0}^{m} a_{ij} x_1^i x_2^{m-i}, & u'_j &= \sum_{i=0}^{m} a'_{ij} x_1^i x_2^{m-i}, \\ v_k &= \sum_{i=0}^{n} b_{ik} x_3^i x_4^{n-i}, & v'_k &= \sum_{i=0}^{n} b'_{ik} x_3^i x_4^{n-i}, \\ & & (c_{jk} x) = \sum_{p,q=1}^{4} c_{jk,pq} x_p x_q \, , \end{aligned} \tag{3}$$

and so on.

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