LACUNARY POWER SERIES AND PEANO CURVES

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Introduction. The existence of a pair of continuous functions $F(t)$, $G(t)$, such that the curve $x = F(t)$, $y = G(t)$ fills completely a certain square is classical. Such a curve will, as usual, be called a Peano curve. The present paper has its origin in the following question: does there exist a Peano curve $x = F(t)$, $y = G(t)$ such that the components $F(t)$, $G(t)$ are respectively the real and the imaginary parts of the values taken by a power series on its circle of convergence? In other words: does there exist a power series $f(z) = \sum_0^\infty a_n z^n$ with radius of convergence equal to 1, continuous in the closed circle $|z| \leq 1$ and such that the values $f(e^{i\theta})$ taken by $f(z)$ on the circumference $|z| = 1$ fill completely a certain square?

We will show that the answer is affirmative. But more interesting seems to be the fact that the above described property is a general property of every lacunary power series $\sum_1^\infty a_k z^{n_k}$ $(n_{k+1}/n_k \geq \lambda > 1)$, with $\sum_1^\infty |a_k| < \infty$, provided that λ is larger than a certain absolute constant, and provided the convergence of $\sum_1^\infty |a_k|$ is slow enough. For instance the reader of the proof of our theorem will verify that the values for $|z| = 1$ of the lacunary series of the Weierstrass type $\sum_1^\infty b^\nu z^{a^\nu}$ where a is an integer, say ≥ 100 and b is a positive number less than 1 but larger, say, than 4/5, form a Peano curve.

We shall prove the following theorem:

Theorem. *There exists an absolute constant λ_0 having the following property. Let $f(z) = \sum_1^\infty a_k z^{n_k}$ be a lacunary power series such that $n_{k+1}/n_k \geq \lambda > \lambda_0$ and that $\sum_1^\infty |a_k| < \infty$. If the convergence of $\sum |a_k|$ is slow enough so that for all p*

$$\frac{\lambda |a_1| + \lambda^2 |a_2| + \cdots + \lambda^p |a_p|}{\lambda^p} < c(\lambda)[|a_{p+1}| + |a_{p+2}| + \cdots],$$

where $c(\lambda)$ is a certain constant depending on λ only, then the values taken by $f(z)$ for $|z| = 1$ fill completely a certain square.

It is obvious that *some* restriction about the rapidity of the convergence of $\sum_1^\infty |a_k|$ is indispensable; for in the above example, e.g., if $b < 1/a$, we would have boundary values with continuous derivatives.

On the contrary the necessity of having $\lambda > \lambda_0 > 1$ and not merely $\lambda > 1$ is not at all obvious: as a matter of fact the problem of the best possible value of λ_0 (whether 1 or larger than 1) remains open. It would be interesting to investigate what is the lower bound of the constant λ_0 such that the theorem (or an analogous theorem) is true.

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