THE THIRTY-NINE SYSTEMS OF QUATERNIONS WITH A POSITIVE NORM-FORM AND SATISFACTORY FACTORABILITY

By Christine S. Williams and G. Pall

1. **Introduction.** The quaternion arithmetics which we shall set forth are of particular interest because for them, and only for them, among systems of rational generalized quaternions with positive definite norm-forms, is factorization always possible and unique, under conditions rather like those for integral Hamiltonian quaternions [5]. It may be surmised therefore that these systems will be susceptible of many applications, and it is our purpose here to tabulate the facts about the individual 39 systems in order to facilitate their use. The systems themselves, and the proof of their unique properties in §5, were derived by Pall [5].

2. **Definitions and notations.** Quaternions (generalized) [4; §3] are quantities of the form $t = t_0 + i_1t_1 + i_2t_2 + i_3t_3$ where the coordinates t_i range over some field, say that of reals, and the basal elements $1, i_1, i_2, i_3$ satisfy a multiplication table associated as follows with a given symmetric ternary matrix $(a_{\alpha\beta})$ and the adjoint matrix $(A_{\alpha\beta})$:

$$i_\alpha^2 = -A_{\alpha\alpha}\ (\alpha = 1, 2, 3); \qquad i_\alpha i_\beta = -A_{\alpha\beta} + \sum_{\delta=1}^{3} a_{\gamma\delta} i_\delta ,$$

$$i_\beta i_\alpha = -A_{\beta\alpha} - \sum_{\delta} a_{\gamma\delta} i_\delta ,$$

α, β, γ being a cyclic permutation of 1, 2, 3. The fundamental number of the system is $d = 4 \mid a_{\alpha\beta} \mid$ and is assumed not zero. The case $(a_{\alpha\beta}) = I$, the identity matrix, gives the Hamiltonian quaternions.

A suitable basis for integral elements [5; §3] is given, in case the $a_{\alpha\alpha}$ and $2a_{\alpha\beta}$ are rational integers, by the quantities $1, j_1, j_2, j_3$, where

$$j_\alpha = i_\alpha + \tfrac{1}{2}\epsilon_\alpha \qquad (\alpha = 1, 2, 3);$$

and $\epsilon_\alpha = 0$ if $2a_{\beta\gamma}$ is even, $\epsilon_\alpha = 1$ if $2a_{\beta\gamma}$ is odd. Thus $t = t_0 + \sum i_\alpha t_\alpha = t_0' + \sum j_\alpha t_\alpha$ (whence $t_0' = t_0 - \frac{1}{2}\sum \epsilon_\alpha t_\alpha$) is integral if and only if t_1, t_2, t_3, and $t_0 - \frac{1}{2}\sum \epsilon_\alpha t_\alpha$ are integers.

It is assumed in this article that the form $f = \sum a_{\alpha\beta}x_\alpha x_\beta$ has integral coefficients $a_{\alpha\alpha}$ and $2a_{\alpha\beta}$, and is positive definite; $\mathfrak{F}$ denotes the adjoint form, of matrix $(A_{\alpha\beta})$, whose elements are in general not integers. However, the quaternary form $F(t_0', t_1, t_2, t_3) = (t_0' + \frac{1}{2}\sum \epsilon_\alpha t_\alpha)^2 + \sum A_{\alpha\beta}t_\alpha t_\beta$ has integral co-

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