AN INTEGRAL EQUATION RELATED TO BESSEL FUNCTIONS

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1. **Introduction.** A study of the integral equation,

$$2 \int_0^\infty \exp(-\alpha \mid s - t \mid - 2t) f(t)\, dt = \lambda f(s) \qquad (\alpha > 0) \tag{1.1}$$

provides a rapid way of obtaining results concerning Bessel functions. This equation arose in the researches of Professor Mark Kac on random noise in radio receivers.

The general procedure will be to examine the kernel and, after solving the equation, use the properties of its corresponding eigenvalues and eigenfunctions. It will be shown that the kernel is positive definite and that the solutions of the equation are Bessel functions. This information and a few theorems from the theory of positive definite kernels, notably Mercer's theorem, lead with great simplicity and rapidity to the proof that the sum of the reciprocals of the squares of the roots of $J_p(z)$ is equal to $[4(p + 1)]^{-1}$ and to the proof that $J_p(z)$ has no complex roots, where in each case $p > -1$. With equal rapidity the positive definiteness of the kernel leads to the completeness in L_2 over (0, 1) of the set $\{J_\alpha(r_n z)\}$ $[\alpha > 0; J_{\alpha-1}(r_n) = 0; r_n > 0, n = 1, 2, 3, \cdots]$, and to the completeness in C over (0, 1) of the set $\{J_p(r_n z)\}$ $[p > -1; J_p(r_n)' = 0; r_n > 0, n = 1, 2, 3, \cdots]$. It is also possible to prove that the set $\{1, J_0(r_n z)\}$ $[J_1(r_n) = 0; r_n > 0, n = 1, 2, 3, \cdots]$ is complete in C over (0, 1). The author thanks Professor W. A. Hurwitz for suggesting this latter proof. The use of the completeness of a kernel to prove the completeness of a set (in C) was first employed by Kneser [3; §32] but, in our opinion, in a more laborious manner due to a less convenient kernel. It may be worth mentioning that we found it rather curious that our integral equation, which arose in a perfectly natural way in a physical problem, led directly to the set $\{J_\alpha(r_n z)\}$ $[\alpha > 0; J_{\alpha-1}(r_n) = 0; r_n > 0, n = 1, 2, 3, \cdots]$. In Kneser's treatment one is led to the set $\{J_\alpha(r_n z)\}$ $[\alpha > 0; J_\alpha(r_n) = 0; r_n > 0, n = 1, 2, 3, \cdots]$.

2. **The kernel and its properties.** We may transform the integral equation into one with a symmetrical kernel. If we multiply the integral equation (1.1) by e^{-s} and call

$$e^{-t} f(t) = \phi(t),$$

we get

$$2 \int_0^\infty \exp[-\alpha \mid s - t \mid - (s + t)] \phi(t)\, dt = \lambda \phi(s) \qquad (\alpha > 0) \tag{2.1}$$

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