A PROPERTY OF THE ELLIPTIC MODULAR NET

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1. For positive values of y in

$$\omega_2/\omega_1 = \omega = x + iy,$$

the discriminant of the cubic polynomial

$$(dp/dw)^2 = 4p^3 - g_2 p - g_3$$

in $p = p(w; \omega_1, \omega_2)$ is

$$\Delta(\omega_1, \omega_2) = g_2^3 - 27g_3^2 = \omega_1^{-12}\Delta(1, \omega),$$

where, if $m_1, m_2 = 0, \pm 1, \pm 2, \cdots$, but $(m_1, m_2) \neq (0, 0)$,

$$g_2(\omega_1, \omega_2) = 60 \sum{}' (m_1\omega_1 + m_2\omega_2)^{-4}, \qquad g_3(\omega_1, \omega_2) = 140 \sum{}' (m_1\omega_1 + m_2\omega_2)^{-6}.$$

Clearly, the function

$$\Delta(\omega) = \Delta(1, \omega)$$

is regular in the half-plane $y > 0$ and is relatively invariant under every substitution $\omega \to S\omega$ of the modul group. Since $\Delta(\omega) \neq$ const., this implies that the x-axis is a natural boundary of $\Delta(\omega)$. In addition, the Eisenstein series g_2, g_3 exhibit for $\Delta(\omega)$ a formal pole at every rational x.

However, it turns out that, *if a certain x-set Z of measure* 0 *is discarded,* $\Delta(\omega) = \Delta(x + iy)$ *tends to a finite, non-vanishing limit as* $y \to +0$. If $\Delta(x)$ denotes this radial limit, it is clear that $\Delta(x)$ is a measurable function which is relatively invariant under every substitution $x \to Sx$ of the modul group, for almost all x. In particular, $\Delta(x)$ is a periodic function, of period 1.

The exclusion of a zero set Z is essential. In fact, the formal poles necessitate that the radial limit $\Delta(x)$ is infinite on a dense sequence of x-values. It follows therefore from well-known general results of Borel and Baire concerning families of continuous functions, that the radial limit $\Delta(x)$ is infinite on an x-set which is of the second category, hence of the power of the continuum, on every x-interval. (Incidentally, there are in Z points x at which the radial limit $\Delta(x)$ fails to exist even if ∞ is allowed as a value.) However, the logarithm of the periodic, measurable function $\Delta(x)$ turns out to be integrable and, in fact, to be of class (L^p) for every p. It is understood that, since $\Delta(\omega) = \Delta(x + iy)$ is known to be distinct from 0 for $y > 0$, the logarithm of $\Delta(x)$ can be defined as the limit of a continuous determination of $\log \Delta(\omega)$.

Finally, the boundary function $\Delta(x)$ proves to exist, for almost all x, not only in the radial sense but also as a Stolzian limit. In other words, *if the real number*

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