SUBDIRECTLY IRREDUCIBLE COMMUTATIVE RINGS

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1. **Introduction.** If S_i $(i = 1, 2, \cdots)$ is any set of rings, the *direct sum* of these rings is the ring of all symbols

$$a = (a_1, a_2, \cdots)$$

with $a_i \varepsilon S_i$ and addition and multiplication defined as follows:

$$(a_1, a_2, \cdots) + (b_1, b_2, \cdots) = (a_1 + b_1, a_2 + b_2, \cdots),$$

$$(a_1, a_2, \cdots)(b_1, b_2, \cdots) = (a_1b_1, a_2b_2, \cdots).$$

Although the notation may suggest a denumerable set of S_i, the concept does not depend on the cardinal number of the set and we therefore do not restrict this cardinal number.

If S is a subring of a direct sum of the rings S_i, then the correspondence

$$a \rightarrow a_i \tag{1}$$

defines a homomorphism of S with a subring of S_i. If, for each i, every element of S_i is the image of some element of S, that is, if the homomorphism (1) is actually a homomorphism of S with the entire ring S_i, S may be said to be the *subdirect sum* of the rings S_i. Thus, if R is isomorphic to the subdirect sum of rings S_i, each S_i is a homomorphic image of R and hence $S_i \cong R/M_i$, where M_i is the ideal whose elements are the elements of R which correspond to the zero element of S_i. Since R is assumed to be isomorphic to the subdirect sum of rings S_i, different elements of R must correspond to different elements of this direct sum, hence the ideals M_i have zero intersection. Conversely, if there exists a set of ideals M_i in R, with zero intersection, R is isomorphic to a subdirect sum of rings R/M_i. (See [1], [3], and other references in [3].)

Following Birkhoff [1], we say the ring R is *subdirectly irreducible* if in any representation of R as a subdirect sum of rings S_i, the homomorphism of R with S_i is actually an isomorphism for at least one i. Thus, R is subdirectly irreducible if, and only if, the intersection of all proper ideals (i.e., ideals different from zero) is itself a proper ideal in R.

As a special case of a theorem of Birkhoff [1; 765], we have the fundamental

THEOREM. *Every ring is isomorphic to a subdirect sum of subdirectly irreducible rings.*

It has also been proved [1; 767] that a *subdirectly irreducible commutative ring without nilpotent elements is a field.* The theorem stated above suggests the desirability of characterizing all subdirectly irreducible rings, and the main purpose of the present paper is to do this for the case of commutative rings.

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