THE ABSOLUTE SUMMABILITY OF POWER SERIES

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1. Let

$$(1.1) \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

be a pair of analytic functions regular for $r = | z | < 1$. Hardy and Littlewood [2] call

$$h(z) = P(f, g) = \sum a_n b_n z^n$$

the "Faltung" or "Parseval function" of $f(z)$ and $g(z)$ and have investigated various problems concerning the summability of the series

$$(1.2) \qquad \sum a_n b_n e^{ni\theta}.$$

An interesting result related to absolute convergence of (1.2) is also given in their paper [2]. It runs:

If
(i) $0 < k \leq 1, pk \geq 1$ (*so that* $\lambda = p/(p + pk - 1) \leq 1$ *and* $p > 1$ *if* $k < 1$);
(ii) f *belongs to* L^λ;
(iii) g *belongs to* $\mathrm{Lip}(k, p)$;
(iv) $p \leq 2$;
then (1.2) *is absolutely convergent.*

The theorem fails in the case $p > 2$, as shown by the example:

$$(1.3) \qquad f(z) = \sum \frac{n^\beta z^n}{(\log n)^\gamma}, \qquad g(z) = \sum \frac{e^{\alpha i n \log n} z^n}{n^{k+\frac{1}{2}}},$$

with

$$\alpha > 0, \qquad 0 < k < 1, \qquad p > 2, \qquad \beta = k - \frac{1}{p}, \qquad \gamma > \frac{1}{\lambda}.$$

We have $f \in L^\lambda$ and $g \in \mathrm{Lip}\, k$. But $\sum | a_n b_n |$ is divergent.

It is, however, interesting to know what can be deduced in the case $p > 2$, when we replace the condition (iii), or its equivalent form

$$M_p(g') = M_p(r, g') = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} | g'(re^{i\theta}) |^p \, d\theta\right)^{1/p} = O((1 - r)^{-1+k})$$

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