THE IDEMPOTENT ELEMENTS OF A COMMUTATIVE RING FORM A BOOLEAN ALGEBRA; RING-DUALITY AND TRANSFORMATION THEORY

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1. **Introduction.** In a previous communication [1] it was shown that the classical symmetry and duality of Boolean Algebra (= Boolean Ring with unit) is merely an instance of a certain symmetry-duality theory inherent in the general concept of ring.

The present paper is mainly concerned with (a) an application of this theory to yield an extension of Stone's Theorem on the characterization of Boolean Rings (see end of this section), (b) a specialization of this extension to the ring of residues (mod m) and, more generally, to the factor rings of principal ideal rings, (c) a remark on Boolean-like rings, that is, a certain generalization of Boolean Rings in which the ring addition has the same formal definition as in ordinary Boolean Rings, and (d) a brief location of the ring-duality theory of [1] within the framework of a more general transformation theory.

To facilitate reference and orientation we recall, in very abbreviated form, certain basic portions of [1]. Let $R = (R, +, \times; 0, 1)$ be a commutative ring with unit, 1, and 0 as its zero element. The elements 0 and 1 are defined as *dual*. Similarly, $\times$, $\otimes$; $+$, $\oplus$; $-$, $\ominus$ constitute respective *dual pairs* of operations; and the operation of *ring-complementation*, *, is *self-dual*. Here

$$(1)\qquad \begin{aligned} a \otimes b &= a + b - ab \qquad\qquad (\text{where } ab = a \times b)\\ a \oplus b &= a + b - 1\\ a \ominus b &= a - b + 1\\ a^* &= 1 - a. \end{aligned}$$

(Notation: Where reading is thereby substantially simplified it is desirable, as in [1], to write $a \,\Delta\, b$ in place of the more systematic notation $a \otimes b$.) Furthermore, for *double-unity* elements, a, (that is, such as have an inverse, a', with respect to $\times$ and also an inverse, a°, with respect to $\otimes$, as will for example be the case for all elements $a \neq 0, \neq 1$ of a field), $'$ and $^\circ$ are *dual*, where

$$a^\circ = a(a - 1)'.$$

One version of the duality theorem then states:

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