CONVERGENCE FACTORS FOR GENERALIZED INTEGRALS

By R. L. Jeffery and D. S. Miller

1. **Introduction.** In a generalization of the derivative of a function J. C. Burkill [1] made use of what he called the Cesàro mean of a function $F(x)$ on an interval defined by the points x and $x + h$. For positive integral values of r these means are given by the relation

$$C_r(F, x, x + h) = \frac{r}{h^r} \int_x^{x+h} (x + h - t)^{r-1} F(t)\, dt, \tag{1}$$

where the integral exists in some suitably defined sense. The upper and lower Cesàro derivatives, $C_r D^* F$ and $C_r D_* F$ are then respectively the upper and lower limits as h tends to zero of the ratio

$$\frac{C_r(F, x, x + h) - F(x)}{h/(r + 1)}.$$

If for a point x these two limits are equal, the function $F(x)$ has a Cesàro derivative at the point x. For the first order mean on an interval (a, b)

$$C_1(F, a, b) = \frac{1}{b - a} \int_a^b F(t)\, dt = \frac{1}{a - b} \int_b^a F(t)\, dt = C_1(F, b, a).$$

In a recent paper [3] this fact entered in a fundamental way in the proofs of results concerning the inversion of Cesàro derivatives of the first order. For $r > 1$ the means $C_r(F, a, b)$ and $C_r(F, b, a)$ are defined respectively by

$$\frac{r}{(b - a)^r} \int_a^b (b - t)^{r-1} F(t)\, dt \text{ and } \frac{r}{(a - b)^r} \int_b^a (a - t)^{r-1} F(t)\, dt,$$

and these are not in general equal. Consequently, the methods used for inverting derivatives of the first order could not be used for derivatives of a higher order, and this led to the following consideration. In (1) the expression $(x + h - t)^{r-1}$ serves as a convergence factor for the integral. Would it be possible to replace this by a convergence factor defined in such a way that the means corresponding to $C_r(F, a, b)$ and $C_r(F, b, a)$ are equal? The present paper is for the most part an answer to this question. It turns out that there is a wide class of suitable factors. The results obtained are parallel to those of Burkill, but owing to the general form of the convergence factors it was necessary to devise new methods of proof.

2. **Definitions and notation.** The symbol $l(x, h)$ will be used to denote the set of points in the interval defined by the points x and $x + h$, whether h be

Received August 12, 1944. The time for this study was made available by a grant from the Carnegie Corporation to Acadia University.