CONTRIBUTIONS TO THE THEORY OF DERIVATES

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Introduction. The derivate numbers of a real-valued function of a real variable may be defined in terms of the related concepts of least upper and greatest lower bounds, and limits of sets of numbers. Among others, A. Denjoy [4], Mrs. G. C. Young [12], U. S. Haslam-Jones [5], A. Khintchine [9], and S. Saks [10] have considered various phases of the problem of derivates of real-valued functions. This paper will consider "derivates" defined for single-valued finite functions $f(x)$, $0 \leq x \leq 1$, with values in an abstract space having certain lattice properties.

A particular space which satisfies the axioms stated in §1 is the space (S) of Lebesgue measurable functions, modulo null functions, defined on the closed unit interval; in §2 we consider properties of derivates of functions having values in this space. In §3 and thereafter we assume that Y satisfies an additional "regularity" axiom. Some results for derivates of semi-continuous functions are obtained in §3. In §4 there is introduced a classification of functions, whereby for a bounded function in a particular class, an upper bound is secured for the class to which the derivates belong. In §5 the distribution of values of derivates of convex functions is investigated.

1. We will assume that Y is a linear semi-ordered space in the sense of Kantorovitch [8]; Y is extended by the addition of infinite elements with the usual properties. In this space the notion of limit is introduced in the following manner: for $\{y_n\}$ a sequence in Y, let

$$\overline{\lim_{n \to \infty}}\, y_n = \inf_n \{\sup (y_n , y_{n+1} , \cdots)\}$$

and

$$\underline{\lim}_{n \to \infty}\, y_n = \sup_n \{\inf (y_n , y_{n+1} , \cdots)\}.$$

These are always defined, but they may be infinite. If

$$\overline{\lim_{n \to \infty}}\, y_n = \underline{\lim}_{n \to \infty}\, y_n ,$$

we say that the sequence $\{y_n\}$ converges, and its limit value is

$$\lim_{n \to \infty} y_n = \overline{\lim_{n \to \infty}}\, y_n = \underline{\lim}_{n \to \infty}\, y_n .$$

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