STRONG SUMMABILITY OF FOURIER SERIES

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1. Let $S_n(x) = \frac{1}{2}a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)$ be a partial sum of the Fourier series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \tag{1.1}$$

of a periodic integrable function with period 2π at $t = x$. The Fourier series (1.1) is said to be strong summable of order 2, or summable H_2, to sum S at $t = x$, provided that

$$\sum_{\nu=0}^{n} | S_\nu(x) - S |^2 = o(n) \qquad (n \to \infty).$$

Now we write $\phi(t) = \frac{1}{2}\{f(x + t) + f(x - t) - 2S\}$.

Concerning this kind of summability, Hardy and Littlewood [1] have given many interesting results and proposed in their course of investigations many questions. Some of these questions were solved by Zygmund and Marcinkiewiez [3]. The remaining are still left open. The object of this paper is to prove the following theorem which gives a solution of one of the remaining questions [1].

Theorem. *If*

$$\int_0^t | \phi(u) | \{1 + \log^+ | \phi(u) |\} \, du = o(t) \qquad (t \to 0), \tag{1.2}$$

then the Fourier series (1.1) *is summable* H_2 *to the sum* S *at* $t = x$.

2. In order to prove this theorem we require several lemmas. We use A or O as an absolute constant different in different occurrences.

Lemma 1. *If*

$$\int_0^t | \phi(u) | \, du = o(t) \qquad (t \to 0), \tag{2.1}$$

then

$$\sum_{\nu=0}^{n} | S_\nu(x) - S |^2 = \frac{8}{\pi^2} \int_{1/n}^{\delta} \frac{\phi(t)}{t^2} \, dt \int_{1/n}^{t} \phi(u) \frac{\sin n(u - t)}{u - t} \, du + o(n).$$

Proof. Now by (2.1) we get

$$S_\nu(x) - S = \frac{2}{\pi} \int_{1/n}^{\delta} \phi(t) \frac{\sin \nu t}{t} \, dt + o(1) \qquad (\nu \leq n).$$

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