SMOOTH FUNCTIONS

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1. **Introduction.** This section is devoted to stating the main results. Proofs and additional details will be given in subsequent sections.

A function $F(x)$ defined in the neighborhood of a point x_0 is said to be *smooth* at that point, if

$$\frac{F(x_0 + h) + F(x_0 - h) - 2F(x_0)}{h} \to 0 \qquad (h \to 0). \tag{1.1}$$

If the derivative $F'(x_0)$ exists and is finite, the function F is smooth at the point x_0, but, of course, the smoothness of F at x_0 does not imply the differentiability of F there. The only thing that follows from (1.1) is that, if the right and the left derivatives of F exist at the point x_0, these derivatives must be equal, so that F is differentiable at x_0. The curve $y = F(x)$ has then no angular point at x_0, and this is the origin of the terminology.

If $F(x)$ is smooth at every point of the (open or closed) interval (a, b), we shall say that F is smooth in (a, b).

If $F(x)$ is smooth at a point x_0, we shall sometimes say that F satisfies condition λ at that point; similarly for smoothness in an interval. If F satisfies the condition

$$\frac{F(x_0 + h) + F(x_0 - h) - 2F(x_0)}{h} = O(1) \qquad (h \to 0), \tag{1.2}$$

we shall say that F satisfies condition Λ at x_0. Similarly we define condition Λ in an interval (a, b). If condition (1.1) or (1.2) is satisfied uniformly in an interval (a, b), and if F is continuous there, we shall say that F satisfies there condition $\lambda^\star$, or condition $\Lambda^\star$. This presupposes that F is defined in some interval comprising (a, b). We shall also write $F \in \lambda^\star$, or $F \in \Lambda^\star$, as the case may be.

Finally, let us suppose that F is periodic, of period 2π, and that $F \in L^p$, where p is any number not less than 1. We shall say that F satisfies condition λ_p, or that $F \in \lambda_p$, if

$$\left\{\int_0^{2\pi} | F(x + h) + F(x - h) - 2F(x) |^p \, dx\right\}^{1/p} = o(h) \qquad (h \to 0). \tag{1.3}$$

Replacing here o by O, we get condition Λ_p.

The notion of smoothness was first considered by Riemann in his classical paper on trigonometric series. He showed that if the coefficients of the trigonometric series

$$\tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

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