AN ANALOGUE OF EULER'S φ-FUNCTION

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Introduction. Let k be a fixed positive integer and consider the k classes of numbers, mod k, of the form $mk + l$, where $m = 0, 1, 2, \cdots$ and $l = 0, 1, 2, \cdots, k - 1$.

If φ is Euler's function, then exactly $\varphi(k)$ of these progressions contain infinitely many primes. If this theorem of Dirichlet is considered as the definition of the function φ, it is natural to introduce an arithmetical function $\rho(k)$ which is similar to φ and is defined as follows:

$\rho(k)$ is a number such that exactly $\rho(k)$ of the k arithmetical progressions contain infinitely many square-free numbers.

The present note concerns the function $\rho(k)$. First, the multiplicative character of φ is paralleled by the fact that $\rho(k)$ *is multiplicative*, i.e., such that $\rho(k_1)\rho(k_2) = \rho(k_1k_2)$, if $(k_1, k_2) = 1$. Hence it is sufficient to determine the values of $\rho(k)$ for $k = p^n$, p a prime and n a positive integer. These values prove to be

$$\rho(p) = p; \qquad \rho(p^n) = p^n - p^{n-2}, \text{ if } n > 1.$$

Consequently, the explicit form of ρ is

$$\rho(k) = k \prod_{p^2 \mid k} (1 - p^{-2}),$$

where it is understood that the product denotes 1 when it is vacuous, i.e., when k has no multiple prime factor. This parallels the representation

$$\varphi(k) = k \prod_{p \mid k} (1 - p^{-1}).$$

While the asymptotic mean value of $\varphi(k)/k$ is known to be $6/\pi^2$, that of $\rho(k)/k$ proves to be $90/\pi^4$. However, there is a substantial difference in the asymptotic behavior of the two functions, in that $\rho(k)/k$ is uniformly almost periodic in the sense of Bohr, while this is not true of $\varphi(k)/k$. The Fourier series of $\rho(k)/k$, obtained from the above product formula, leads to an explicit trigonometrical representation of $\rho(k)/k$ in terms of Ramanujan sums.

1. First, for fixed positive l, the arithmetical progression $mk + l$, where m varies, contains an infinity of square-free integers if and only if (k, l) is square-free. In fact, let d denote this greatest common divisor. Let h and j denote that unique pair of positive integers for which $k = dh$, $l = dj$ and $(h, j) = 1$. Then $mk + l = d(mh + j)$. Hence $mk + l$ is square-free for no m, if d is not square-free. If, on the other hand, d is square-free, then $mk + l$ is square-free

Received June 26, 1944.