THE LEBESGUE CONSTANTS OF MÖBIUS' INVERSION

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Introduction. If either of two arbitrary functions, say $f = f(n)$ and $f' = f'(n)$, of the positive integer n is given, the other function is uniquely determined by the assignment

(1) $$f(n) = \sum_{d|n} f'(d),$$

since this linear transformation of f' into f has the unique inversion

(2) $$f'(n) = \sum_{d|n} \mu(n/d)f(d).$$

It is understood that the summation index d (or, equivalently, the quotient n/d) runs through all divisors $d(\geq 1)$ of n, and that $\mu(m)$ denotes Möbius' function.

A formal principle, first applied by Gauss, can be formulated as follows (for references to the classical literature see [3; 9–10]). The connection assigned by (1) or (2) for arbitrary function mates f, f' is such that

(3) $$M(f) = \sum f'(n)/n$$

holds if either the asymptotic average

(4) $$M(f) = \lim_{n\to\infty} \frac{f(1) + f(2) + \cdots + f(n)}{n}$$

exists (as a *finite* limit) or the series $\sum f'(n)/n$, where

(5) $$\sum g(n) = \sum_{n=1}^{\infty} g(n),$$

is convergent. In fact, if $[x]$ denotes the greatest integer not exceeding x, it is clear from (1) that

$$f(1) + f(2) + \cdots + f(n) = [n/1]f'(1) + [n/2]f'(2) + \cdots + [n/n]f'(n).$$

Since $[n/m]$ is 0 for every $m > n$, this is equivalent to

$$\{f(1) + f(2) + \cdots + f(n)\}/n = \sum_{m=1}^{\infty} [n/m]f'(m)/n.$$

Since the m-th term of the last series tends to $f'(m)/m$ *if* m *is fixed and* $n \to \infty$, the relation (3) follows if the limit process $n \to \infty$ is applied term-by-term.

However, the legitimacy of this formal passage to the limit is by no means evident. In this regard, it is revealing to consider the particular function $f(n)$

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