CRITERIA FOR COMPLETENESS OF ORTHONORMAL SETS AND SUMMABILITY OF FOURIER SERIES

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1. **Introduction.** It is the object of this paper to generalize and systematize methods that have been found to be useful for determining whether a given orthonormal set is complete. The methods depend upon use of kernels $K(x, y, t)$ determined by the orthonormal set and a convergence-factor method G of summability. Several necessary and sufficient conditions for completeness are given. These criteria apply to all orthonormal sets of functions $\phi_n(x)$, bounded or unbounded, defined over entire Euclidean spaces or any measurable subsets of them. When G is restricted to a wide class containing all familiar methods of summability (convergence, Cesàro, Abel, Euler, Borel, LeRoy, etc.) as well as many non-regular methods, the criteria are independent of the G used to determine the kernel $K(x, y, t)$. If one method of summability is, in a particular situation, better than another, it is not because the one kernel has pertinent properties the other does not have; it is because it is easier to discover the pertinent properties of the one kernel than of the other. That this can be true is a consequence of the fact that Fourier series of functions in L_2, even though they need not be everywhere convergent and may be defined over sets of infinite measure, are in many respects as well behaved and as manageable as series converging uniformly over sets of finite measure. Several of the fundamental ideas are exhibited by Wiener's development [7; 51–66] of properties of the Hermite orthonormal set and proof of completeness of the set. For general discussions of orthonormal sets, see [1], [4], and [8]. For extensive references, see [3].

Terminology, notation, and fundamental facts about completeness to be used later are given in §2. A general form of the Riesz-Fischer theorem is given in §3. Convergence factors $G_n(t)$, defining a method of summability G not necessarily regular, are introduced and used to define $K(x, y, t)$ heuristically in §4. The kernels $K(x, y, t)$ are defined and associated with the problem of completeness in §5. The rôles of step functions and continuous functions are presented briefly in §6 and §7. Properties of $K(x, y, t)$, x fixed, are given in §8. Criteria for completeness involving convergence in mean and summability, respectively, are given in §9 and §10. Use of the simplest of the criteria for completeness is illustrated in §11 by a simple proof of the well-known fact that the Hermite orthonormal set is complete.

2. **Complete sets.** Let E be Euclidean space of one or more dimensions or a measurable subset of one of these spaces. The single symbols x and y will be

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