THE CONVERGENCE OF SEQUENCES OF HADAMARD DETERMINANTS

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1. **Introduction.** Suppose the function $f(z)$ represented by its Maclaurin series $\sum_n c_n z^n$ is regular in the closed circle C except for the p poles $z_1, z_2, \cdots, z_p$ inside and the q poles $z_{p+1}, z_{p+2}, \cdots, z_{p+q}$ on this circle (the poles being counted according to their multiplicities). It is well known that

$$\limsup_{n\to\infty} | d_n^{(m)} |^{-1/n} = | z_1 z_2 \cdots z_m | \qquad (m = p, p+1, \cdots, p+q),$$

where $d_n^{(m)}$ is the determinant of m-th order

$$d_n^{(m)} = \begin{vmatrix} c_{n+1} & c_{n+2} & \cdots & c_{n+m} \\ c_{n+2} & c_{n+3} & \cdots & c_{n+m+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n+m} & c_{n+m+1} & \cdots & c_{n+2m-1} \end{vmatrix} \qquad (n = 1, 2, \cdots).$$

It was proved by Hadamard [2], [4] that in the case $m = p + q$ the limit superior of the above equation is a regular limit, and moreover

$$\lim_{n\to\infty} d_n^{(p+q)}/d_{n+1}^{(p+q)} = z_1 z_2 \cdots z_{p+q} .$$

In a previous paper [1; §4] I have shown that $\lim_n d_n^{(m)}/d_{n+1}^{(m)}$ exists also if $m = p + \kappa$, where κ is the number of poles of highest order (each counted as one) on C. In many cases there are other numbers m between p and $p + q$ for which this limit exists, but it is clear that it does not exist for all such numbers. In this paper we show that the convergence does not, in general, depend on the relative position of the poles, but only on their multiplicities, and we obtain a complete characterization of the numbers m for which there is convergence in the criterion that the sequence of determinant quotients converges, in general, if and only if a certain Diophantine maximum problem (see §4) has a unique solution.

To prove this result it did not suffice to have an estimate of the determinant $d_n^{(m)}$, but it became necessary to have an evaluation of its principal part (as a function of n). This is done in several steps, the results of which are stated in the lemmas of §§2-4. The rest of the paper is devoted to the discussion of the Diophantine maximum problem.

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