TWO TYPES OF FUNCTION FIELD TRANSCENDENTAL NUMBERS

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Let p denote a fixed prime, $GF(p^n)$ the finite field of order p^n ($n = 1, 2, \cdots$), and Γ the algebraic closure of $GF(p)$. For e indeterminates $x_1, \cdots, x_e$, $\Gamma(x_1, \cdots, x_e)$ will denote the field of rational functions in $x_1, \cdots, x_e$ with coefficients from Γ. If $E \neq 0$ and $G \neq 0$ are two polynomials in $x_1, \cdots, x_e$ (with coefficients from Γ or, what is the same thing, from some $GF(p^n)$), define $\deg E/G = \deg E - \deg G$, where deg is an abbreviation for degree. If we write $-\deg 0 = \infty$, then $-\deg$ defines an exponential valuation [2] of $\Gamma(x_1, \cdots, x_e)$. Denote by Φ the corresponding completion of $\Gamma(x_1, \cdots, x_e)$. Here we shall consider the transcendence over $\Gamma(x_1, \cdots, x_e)$, or equivalently over $GF(p; x_1, \cdots, x_e)$, of two types of elements of Φ defined by infinite series.

The first is an analogue of real numbers of the form

$$\sum_{k=0}^{\infty} \frac{1}{g^{q^k}},$$

where $g > 1$, $q > 1$ are rational integers. These numbers have been proved transcendental (over the rational number field, of course) by Kempner [1]. The analogous series in Φ, namely

$$\sum_{k=0}^{\infty} \frac{1}{G^{q^k}}, \tag{1}$$

where G is a polynomial in $\Gamma(x_1, \cdots, x_e)$ of $\deg G > 0$ and $q > 1$ a rational integer, is not always transcendental over $GF(p; x_1, \cdots, x_e)$. The result is given by

THEOREM 1. *The series* (1) *is* (i) *algebraic if $q > 1$ is of the form p^s*; (ii) *transcendental otherwise.*

The second series considered is

$$\sum_{k=0}^{\infty} \frac{1}{G^{k^q}} \tag{2}$$

with the same hypotheses on G and q. This is analogous to the real number

$$\sum_{k=0}^{\infty} \frac{1}{g^{k^q}},$$

whose nature does not seem to be known. For (2) we prove

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