THE GONTCHAROFF POLYNOMIALS

By Norman Levinson

1. The Gontcharoff polynomials

$$G_0(z) = 1, \qquad G_1(z, z_0) = z - z_0 ,$$

(1.0)

$$G_n(z, z_0, z_1, \cdots, z_{n-1}) = \int_{z_0}^{z} ds_1 \int_{z_1}^{s_1} ds_2 \cdots \int_{z_{n-1}}^{s_{n-1}} ds_n \qquad (n > 1)$$

are of considerable interest in interpolation theory [3]. In a number of applications the extensiveness of the results obtained depends on the bounds which can be found for these polynomials.

With $|z| \leq 1$, $|z_k| \leq 1$, $k = 0, 1, \cdots$, we denote the maximum value of $|G_n(z, z_0, z_1, \cdots, z_{n-1})|$ by M_n. (Since the G_n are homogeneous of degree n it follows that, with the z's restricted to a circle of radius R, $|G_n| \leq M_n R^n$.)

It has been shown [2] that

$$M_n < A\left(\frac{1}{\log 2}\right)^n = A(1.44 \cdots)^n,$$

where A is some constant. Here we shall show that, for $n > 1$, there exists an r satisfying

$$M_n < r^{n+1}, \qquad r < 1.386. \tag{1.1}$$

It is easy to show that (1.1) leads to an improved value for Whittaker's constant W. W is defined as the least upper bound of numbers c such that, if $f(z)$ is an entire function of exponential type c, and if $f(z)$ and each of its derivatives have at least one zero in the unit circle, then $f(z) \equiv 0$. Boas has recently shown [1] that $W > .718$. It has been conjectured by Pondiczery in a communication to Boas that $W = 2/e = .736$. We shall now show that $W > 1/1.386 > .7215$.

Let $f(z)$ be an entire function of exponential type c. Then, if

$$f(z) = \sum_0^\infty \frac{a_n}{n!} z^n,$$

it follows that $a_n = O(c + \epsilon)^n$, $\epsilon > 0$. That is, for any $b > c$ it follows that for sufficiently large n

$$|a_n| < b^n. \tag{1.2}$$

Denote by z_0, z_1, z_2, etc. the points inside the unit circle where $f(z)$, $f'(z)$, $f''(z)$, etc. vanish. Then clearly

$$f(z) = \int_{z_0}^{z} ds_1 \int_{z_1}^{s_1} ds_2 \cdots \int_{z_{n-1}}^{s_{n-1}} f^n(s)\, ds.$$

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