ADDITIVE PROPERTIES OF COMPACT SETS

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There are a great number of theorems in topology concerning the relationship between the sets A, B, $A \cdot B$, and $A + B$. In this paper a further study of this relationship is made.

It will be assumed throughout that all sets used lie in a compact metric space. All of our complexes and cycles will be non-oriented, and the Vietoris cycles used will consist of these cycles as coördinates. Finally the boundary of the i-dimensional Vietoris cycle x^i will be denoted by $\dot{x}^i$, and $\delta(A)$ will be used to denote the diameter of the set A.

Let p^i be a property concerned with i-cycles. We shall consider properties p^i which satisfy

Theorem A. *If M_1 and M_2 are two compact subsets of a compactum $M_1 + M_2$ having p^i and $M_1 \cdot M_2$ has p^{i-1}, then $M_1 + M_2$ has p^i.*

Definition. A property p^i satisfying Theorem A will be called *additive*.

Theorem 1. *(Uniform) local i-connectedness is additive.*

Proof. Let $\epsilon > 0$ be arbitrary. There exists a positive number δ' corresponding to the local i-connectedness of M_1 and M_2 such that any i-dimensional cycle of M_1 or M_2 of diameter $< 2\delta'$ is ~ 0 in a subset of M_1 or M_2 of diameter $< \frac{1}{3}\epsilon$. Since $M_1 \cdot M_2$ is $(i - 1)$-lc (locally $(i - 1)$-connected), there exists a positive number $\delta < \delta'$ such that any $(i - 1)$-dimensional Vietoris cycles of $M_1 \cdot M_2$ of diameter $< \delta < \frac{1}{3}\epsilon$ is ~ 0 in a subset of $M_1 \cdot M_2$ of diameter $< \delta'$. Let x^i be an i-dimensional Vietoris cycle of $M_1 + M_2$ with diameter $< \delta$. By infinitesimal alterations in x^i we can obtain a Vietoris cycle, which without loss of generality can be considered as x^i itself, such that in almost all the coördinates each simplex is contained wholly in M_1 or M_2 (i.e., no simplex lies partly in $M_1 - M_1 \cdot M_2$ and partly in $M_2 - M_1 \cdot M_2$). For each (sufficiently large) j let x_{1j}^i be the complex consisting of those simplexes of x_j^i which lie entirely in M_1. Let $x_{2j}^i = x_j^i - x_{1j}^i$, which by the above alterations will consist entirely of simplexes contained in M_2 (but not wholly in M_1). Choose $\eta > 0$ arbitrarily. Then, for almost all j, k, $x_j^i \underset{\eta}{\sim} x_k^i$ (in a subset of $M_1 + M_2$ of diameter $< \delta < \frac{1}{3}\epsilon$). Now $\dot{x}_{1j}^i = \dot{x}_{2j}^i = x_j^{i-1}$ is an $(i - 1)$-dimensional cycle of $M_1 \cdot M_2$. A subsequence of (x_j^{i-1}) will give a Vietoris cycle x^{i-1} of $M_1 \cdot M_2$ with diameter $< \delta$. Therefore $x^{i-1} \sim 0$ in a subset of $M_1 \cdot M_2$ of diameter $< \delta'$, i.e., there exists a chain y^i of $M_1 \cdot M_2$ bounded by x^{i-1}. Now $x_{1j}^i + y_j^i$ is a cycle of M_1 and $x_{2j}^i + y_j^i$ is a cycle of M_2. Choosing subsequences we obtain cycles $(x_{1j}^i + y_j^i)$ and $(x_{2j}^i + y_j^i)$ with diameters $< 2\delta'$ in M_1 and M_2 respectively. Therefore $(x_{1j}^i + y_j^i) \sim 0$ in a

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