ARBITRARY POINT TRANSFORMATIONS

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Introduction. We shall be concerned with point correspondences or transformations $\tau: y = f(x)$ which associate with each point $x:(x_1, x_2, \cdots, x_m)$ of Euclidean m-space S_m a point $y:(y_1, y_2, \cdots, y_n)$ of Euclidean n-space S_n. We shall refer to y as the *mate* or *image* of x, to x as *a mate or an inverse* of y. The image of a subset A of S_m will be understood to be the set of mates of the elements of A; the inverse of a subset B of S_n, the set of all inverses of all the elements of B. It is assumed that f is "one-valued", though this restriction is not essential for our considerations or results. Except for this assumption, f will be taken to be unconditioned; in particular, τ is not assumed to be biunique.

There is a fairly considerable literature on properties of unconditioned functions f (see, for example, [7], [9], [1], [5], [6], [2], [3]). Some of these properties refer not so much directly to f as to associated entities, such as the saltus. The more interesting properties in question give direct information concerning the structure of f itself.

In the present paper, we prove a number of theorems on the *structure of an arbitrary point transformation* τ in the sense of the opening paragraph. §1 deals with descriptive and §2 with metric theorems of this nature.

It will be observed that most of the theorems of the present paper may be readily interpreted as expressing *continuity properties* of the point transformation τ.

The principal result (Theorem XI) *characterizes the set of "salient points" of* τ *by means of two simple, geometric properties.* We proceed to explain what is meant by salient point.

Let us depict τ in the Euclidean space $S = S_{m+n}$ of $m + n$ dimensions by representing the mating of y with x by means of the point $z:(x_1, x_2, \cdots, x_m; y_1, y_2, \cdots, y_n)$ of S. We write, for short, (x, y) for the latter point, and shall speak of $(x_1, x_2, \cdots, x_m)$ as the x-coördinate, of $(y_1, y_2, \cdots, y_n)$ as the y-coördinate of z. By the *graph* I of τ, we shall understand the set of points $(x, f(x))$ as x ranges over S_m. By an open (closed) *oriented cylinder* C_{br}, where $b:(b_1, b_2, \cdots, b_n)$ is a point of S_n, and r a positive real number, we understand the set of points of S whose y-coördinate, as point of S_n, is at distance $< r (\leq r)$ from b; C_{br} is said to be rational if r and every b_ν ($\nu = 1, 2, \cdots, n$) are rational. If A is any point set lying in S, we understand by $X(S)$ the set of x-coördinates of the points of A.

In terms of the notions just defined, we define a *salient point* of τ as a point $\zeta:(\xi, \eta)$ of S such that, for every open oriented cylinder C containing ζ, the set $X(IC)$ has positive upper density at ξ (in the sense of *exterior* Lebesgue measure).

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