ALGEBRAIC INTEGERS WHOSE CONJUGATES LIE IN THE UNIT CIRCLE

By Carl Ludwig Siegel

Let θ be an algebraic integer and assume that all conjugates of θ, except θ itself, have an absolute value less than 1. Then $-\theta$ also has this property; on the other hand, θ is real. Without loss of generality we may therefore suppose $\theta \geq 0$. Since the norm of θ is a rational integer, we have $\theta \geq 1$, except for the trivial case $\theta = 0$. Recently R. Salem [1] discovered the interesting theorem that the set S of all θ is closed and that $\theta = 1$ is an isolated point of S. Consequently there exists a smallest $\theta = \theta_1 > 1$. We shall prove that θ_1 is the positive zero of $x^3 - x - 1$ and that also θ_1 is isolated in S. Moreover we shall prove that the next number of S, namely the smallest $\theta = \theta_2 > \theta_1$, is the positive zero of $x^4 - x^3 - 1$ and that θ_2 is again an isolated point of S. Since $\theta_1 = 1.324 \cdots$, $\theta_2 = 1.380 \cdots$, both numbers are less than $2^{\frac{1}{2}}$; therefore our statements are contained in the following:

Theorem. *Let θ be an algebraic integer whose conjugates lie in the interior of the unit circle; if $\pm\theta \neq 0, 1, \theta_1, \theta_2$, then $\theta^2 > 2$.*

It is easily seen that the positive zero θ of each polynomial

$$x^n(x^2 - x - 1) + x^2 - 1 \qquad (n = 1, 2, 3, \cdots),$$

$$x^n - \frac{x^{n+1} - 1}{x^2 - 1} \qquad (n = 3, 5, 7, \cdots),$$

$$x^n - \frac{x^{n-1} - 1}{x - 1} \qquad (n = 3, 5, 7, \cdots)$$

belongs to S; all these numbers θ lie in the interval $1 < x < \theta_0 = \frac{1}{2}(1 + 5^{\frac{1}{2}})$, and θ_0 is their only limit point. I have not been able to decide whether θ_0 is the smallest positive limit point in S.

Let θ be a positive number of the set S and let

$$P(x) = x^m + a_1x^{m-1} + \cdots + a_m$$

be the irreducible polynomial with the zero θ. Define

$$f = f(x) = \frac{\pm P(x)}{x^m P(x^{-1})} = \sum_{n=0}^{\infty} b_n x^n, \tag{1}$$

where the sign is fixed by the condition $b_0 > 0$, i.e., $b_0 = | a_m |$. The coefficients

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