THE BERNSTEIN-WIDDER THEOREM ON COMPLETELY MONOTONIC FUNCTIONS

By Harry Pollard

1. Our purpose here is to give a self-contained proof of the following well-known theorem due to Bernstein and Widder [3].

Theorem. *Let $f(x)$ be a real function such that*

$$f(0) = f(0+), \qquad (-1)^k f^{(k)}(x) \geq 0 \qquad (0 < x < \infty; k = 0, 1, \cdots). \tag{1}$$

Then it admits the representation

$$f(x) = \int_0^\infty e^{-xt}\, d\alpha(t), \tag{2}$$

where $x \geq 0$ and $\alpha(t)$ is increasing and bounded.

2. We shall begin by reproducing Widder's proof [2; 145, 146] that under conditions (1) the limits

$$\lim_{x\to\infty} \frac{(-1)^k}{k!} x^k f^{(k)}(x) = L_k \qquad (k = 0, 1, \cdots) \tag{3}$$

exist. For $x = 0$ the result is clear, since $f(x)$ is non-negative and decreasing. Form the function $f(x) - xf'(x)$. It has the non-positive derivative $-xf''(x)$ and is non-negative for $x > 0$. It therefore approaches a limit as $x \to \infty$, and so $-xf'(x)$ does also. Assume the limits L_k exist for $k = 0, 1, \cdots, n$. Consider

$$F(x) = f(x) - xf'(x) + \frac{x^2 f''(x)}{2!} - \cdots + \frac{(-1)^{n+1}}{(n+1)!} x^{n+1} f^{(n+1)}(x)$$

with derivative

$$(-1)^{n+1} x^{n+1} f^{(n+2)}(x)/(n+1)!.$$

$F(x)$, being non-negative and decreasing, approaches a limit. All its terms except the last have limits by assumption. Hence the last does also. The induction is thus complete.

3. In this section k is a fixed positive integer. By successive integration by parts and application of (3) it is seen that for $x > 0$

$$f(x) - M_k = \frac{(-1)^k}{(k-1)!} \int_0^\infty u^{k-1} f^{(k)}(u + x)\, du, \tag{4}$$

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